COLLOQUIUM MATHEMATICUM

VOL. XXXI 1974 FASC. 1

ON ELEMENTARY MAPS

BY

JERZY DYDAK (WARSZAWA)

Let $f\colon X \to Y$ be a continuous map.

A map f is said to be *elementary* if X is a metric space with metric ϱ and, for every two different points $x, y \in X$ with $f(x) = f(y)$, there exists a positive number a such that $\varrho(x, y) \geqslant a$ (see [1], p. 84).

By an *order* of f we mean the number $\sup_{y \in Y} \operatorname{card} f^{-1}(y)$.

We say that f is *simple* if its order is less than or equal to 2 (see [1], p. 84).

A map f is said to be *k-to-one* if $\operatorname{card} f^{-1}(y) = k$ for every $y \in f(X)$.

A map f is called a *covering projection* if each point $x \in Y$ has an open neighborhood U such that $f^{-1}(U)$ is the disjoint union of open subsets of X each of which is mapped by f homeomorphically onto U (we say that U is *evenly covered* by f; see [3], p. 62).

A *topological n-manifold* (without the boundary) is a paracompact Hausdorff space in which each point has an open neighborhood (called a *coordinate neighborhood* in the manifold) homeomorphic to R^n.

In paper [1] the authors have raised the following problem (Problem 3, p. 92):

Is it true that every elementary map with a compact domain is a superposition of a finite number of elementary simple maps?

The answer to this question is negative. Moreover, there exist elementary maps of arbitrary large finite order which are not a superposition of elementary maps of order less than their order.

THEOREM 1. *Let M and N be compact, non-empty and connected n-manifolds. Then $f\colon M \to N$ is an elementary map iff it is a covering projection.*

Proof. Necessity. Let a be a positive number such that $x, y \in M$, $x \neq y$, and $f(x) = f(y)$ imply $\varrho(x, y) \geqslant a$. For every $x \in M$, let W_x be a coordinate neighborhood of x contained in $K(x, \frac{1}{3}a)$. Since $f|\overline{K(x, \frac{1}{3}a)}$ is an injective map, W_x is mapped by $f|W_x$ homeomorphically onto $f(W_x)$. It follows from Theorem 6-54 of [2] that $f(W_x)$ is an open subset of N. Hence $f(M)$ is an open-closed subset of N, and so, in view of the connectedness of N, it must be $f(M) = N$.

Let $y \in N$. The set $f^{-1}(y)$ is compact and every two different points belonging to $f^{-1}(y)$ have the distance greater than $\frac{1}{2}a$. Hence $f^{-1}(y)$ is finite. Since $f(W_x)$ is an open neighborhood of y for every $x \in f^{-1}(y)$, and $y \notin f\big(M - \bigcup_{x \in f^{-1}(y)} W_x\big)$, the set

$$U = \bigcap_{x \in f^{-1}(y)} f(W_x) - f\big(M - \bigcup_{x \in f^{-1}(y)} W_x\big)$$

is an open neighborhood of y. By the equality $f\big(A \cap f^{-1}(B)\big) = f(A) \cap B$, we have

$$f[f^{-1}(U) \cap \big(M - \bigcup_{x \in f^{-1}(y)} W_x\big)] = U \cap f\big(M - \bigcup_{x \in f^{-1}(y)} W_x\big) = \emptyset.$$

Hence

$$f^{-1}(U) \cap \big(M - \bigcup_{x \in f^{-1}(y)} W_x\big) = \emptyset \quad \text{and} \quad f^{-1}(U) \subset \bigcup_{x \in f^{-1}(y)} W_x.$$

Thus

$$f^{-1}(U) = \bigcup_{x \in f^{-1}(y)} W_x \cap f^{-1}(U),$$

$$f\big(W_x \cap f^{-1}(U)\big) = f(W_x) \cap U = U \quad \text{for } x \in f^{-1}(y),$$

$$W_x \cap W_z = \emptyset \quad \text{for } x \neq z,\ x, z \in f^{-1}(y),$$

and $f \mid W_x$ is a homeomorphism. Hence U is evenly covered by f, and f is a covering projection.

Sufficiency. It follows from Lemma 2.1.8 of [3] that every covering projection is a local homeomorphism. Let us consider an open covering

$$P = \{U \subset M\colon\ U = \operatorname{Int} U \text{ and } f \mid U \text{ is an injective map}\}$$

of M. Let a be a Lebesgue number of the covering P ([2], Theorem 1-32). If $\varrho(x, y) < a$, then there exists a $U \in P$ containing x and y. Thus $f(x) \neq f(y)$ and, for every $z, t \in M$ with $f(z) = f(t)$, $z \neq t$, it follows that $\varrho(z, t) \geqslant a$. Hence f is an elementary map.

THEOREM 2. *Let M and N be compact, non-empty and connected n-manifolds. If $f\colon M \to X$ and $g\colon X \to N$ are elementary maps and $f(M) = X$, then X is an n-manifold.*

Proof. Let a and b be numbers such that $x, y \in M$, $x \neq y$ and $f(x) = f(y)$ imply $\varrho_M(x, y) \geqslant a$, and $z, t \in X$, $z \neq t$ and $g(z) = g(t)$ imply $\varrho_X(z, t) \geqslant b$ (ϱ_M and ϱ_X are metrics on M and X, respectively). Let $c < a$ be a Lebesgue number of the covering $\{f^{-1}[K(x, \frac{1}{2}b)]\colon x \in X\}$. If $x \neq y$, $x, y \in M$ and $\varrho_M(x, y) < c$, then there exists a $z \in X$ with $x, y \in f^{-1}\big(K(z, \frac{1}{2}b)\big)$. Since $\varrho_M(x, y) < c < a$, we have $f(x) \neq f(y)$, $f(x), f(y) \in K(z, \frac{1}{2}b)$. Thus $f(x) \neq f(y)$, $\varrho_X\big(f(x), f(y)\big) < b$ and $gf(x) \neq gf(y)$. Hence gf is an elementary map. By Theorem 1, gf is a covering projection. Every covering projection is an open map. Thus $gf(f^{-1}(U)) = g(U)$ is an open set for every open set $U \subset X$. Hence g is an open map. Since g is an elementary map, for every $x \in X$, there exists an open neighborhood U_x such that $g|U_x$ is an injection. This implies that $g|U_x$ is a homeomorphism onto the n-manifold $g(U_x)$. Hence X is an n-manifold.

Now consider maps $f_n\colon S^1 \to S^1$ defined by $f_n(z) = z^n$. These maps are covering projections (see [3], p. 62) and, by virtue of Theorem 1, they are elementary maps. Suppose

$$f_k = g_l \circ \ldots \circ g_1 \quad \text{for some } k,$$

where $g_1\colon S^1 \to X_1$, $g_2\colon X_1 \to X_2, \ldots, g_l\colon X_{l-1} \to S^1$ are elementary maps. We can assume that $g_i(X_{i-1}) = X_i$ for $1 \leqslant i < l$. For arbitrary $1 \leqslant i < l$, we have

$$g_i \circ \ldots \circ g_1\colon S^1 \to X_i, \qquad g_l \circ \ldots \circ g_{i+1}\colon X_i \to S^1, \qquad g_i \circ \ldots \circ g_1(S^1) = X_i,$$

and $g_i \circ \ldots \circ g_1$, $g_l \circ \ldots \circ g_{i+1}$ are elementary maps. It follows from Theorems 1 and 2 that X_i are 1-manifolds and g_i's are covering projections. Since spaces X_i are connected, there exists, for every $1 \leqslant i \leqslant l$, the number p_i such that g_i is a p_i-to-one map (see [3], 2.2.3 and 2.3.8). Then $k = p_l \ldots p_1$, and maps f_p, where p is a prime number, are not superpositions of elementary maps of order less than p.

REFERENCES

[1] K. Borsuk and R. Molski, *On a class of continuous mappings*, Fundamenta Mathematicae 45 (1958), p. 84-98.

[2] J. G. Hocking and G. S. Young, *Topology*, Reading, Mass., 1961.

[3] E. Spanier, *Algebraic topology*, New York 1966.

Reçu par la Rédaction le 4. 4. 1972