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ON THE PROBABILITY THAT k GENERALIZED INTEGERS ARE RELATIVELY H-PRIME

BY

ŠTEFAN PORUBSKÝ (BRATISLAVA)

In 1900 Lehmer [3] and in 1972 Nymann [4] considered the number $T_k(n)$ of k-tuples of positive integers $[x_1, \ldots, x_k]$ with $1 \leqslant x_i \leqslant n$ and the greatest common divisor $(x_1, \ldots, x_k) = 1$. They proved that

$$T_k(n) = \zeta^{-1}(k)n^k + \begin{cases} O(n\log n) & \text{if } k = 2, \\ O(n^{k-1}) & \text{if } k > 2. \end{cases} \tag{1}$$

In 1976 Benkoski [1] investigated the following generalization of these results. If r is a positive integer, then the r-th power greatest common divisor $(a, b)_r$ of a and b is the largest r-th power that divides both a and b. He proved that the number $T_k^r(n)$ of k-tuples of positive integers $[x_1, \ldots, x_k]$ with $1 \leqslant x_i \leqslant n$ and $(x_1, \ldots, x_k)_r = 1$ satisfies for $rk > 2$, $k \neq 1$, the following relation:

$$T_k^r(n) = \zeta^{-1}(rk)n^k + O(n^{k-1}). \tag{2}$$

This together with the previous results yields

$$\lim_{n\to\infty} T_k^r(n)/n^k = \zeta^{-1}(rk) \tag{3}$$

provided $k \neq 1$, $rk \geqslant 2$.

In the same paper [1] Benkoski considered two related questions. Namely, given a set S of primes, let $\langle S\rangle$ denote the set containing 1 and the all positive integers every prime factor of which lies in S. Similarly as before, let $T_k^r(S, n)$ denote the number of k-tuples $[x_1, \ldots, x_k]$ with $1 \leqslant x_i \leqslant n$ and $(x_1, \ldots, x_k)_r \in \langle S\rangle$. He proved that if S or the complement of S (within the set of primes) is finite, then

$$\lim_{n\to\infty} T_k^r(S, n)/n^k = \prod_{p\notin S} (1-p^{-rk}). \tag{4}$$

In the meantime Nymann [5] has considered a similar complex of questions. He introduced the notion of a divisible set as a set of positive integers which contains the product of two integers only if it contains both factors. It is easy to see that V is divisible if and only if V is generated by the set of primes in V, i.e. if it is of the form $\langle S\rangle$ for some set S of primes. He proved that if V is divisible and

$$V(x) = \sum_{\substack{t\in V\\ t\leqslant x}} 1 = Ax + O(1),$$

then the number $V_k(n)$ of k-tuples of positive integers $[x_1, \ldots, x_k]$ with $1 \leqslant x_i \leqslant n$, $x_i \in V$ and $(x_1, \ldots, x_k) = 1$ satisfies

$$V_k(n) = (An)^k \prod_{p\in V}(1-p^{-k}) + \begin{cases} O(n\log n) & \text{if } k = 2, \\ O(n^{k-1}) & \text{if } k > 3 \end{cases} \tag{5}$$

(p always denotes a prime). Further, Nymann proved that if V is generated by all but a finite number of primes, then

$$\lim_{n\to\infty} V_k(n)/\big(V(n)\big)^k = \prod_{p\in V}(1-p^{-k}). \tag{6}$$

On the other hand, if we remove all but a finite number of primes, then the corresponding limit on the left-hand side of (6) vanishes. In view of this surprising phenomenon, Nymann asks what happens if V is generated by infinitely many primes but also infinitely many primes are excluded. In what follows we give a partial answer to this question.

Nymann and Leahey considered also the question what happens when the integers are chosen according to the binomial distribution. Since we shall not be interested in this direction, we refer the reader to [6] for more details.

Finally, Knopfmacher extended in [2] the original Nymann's result to arithmetical semigroups satisfying the so-called Axiom A. This is the direction which we shall follow. We introduce the notion of arithmetical subsemigroup which enables us to reprove the above-mentioned results via a unified argument and, in a more general setting, to obtain several improvements even for the case of positive integers.

1. Arithmetical semigroups and subsemigroups. Our terminology and notation will be as in Wegmann [7] with the only change that we shall use the more convenient designation "arithmetical semigroup" from [2] instead of Wegmann's "F-Halbgruppe".

Let G be a free Abelian multiplicative semigroup with identity element 1 and a countable set $P(G)$ of generators — called the *primes*. Such a semigroup G will be called an *arithmetical semigroup* if, in addition, there exists a real-valued norm mapping $|\cdot|$ on G such that

(i) $|ab| = |a|\,|b|$ for all $a, b \in G$;

(ii) the total number $G(x)$ of elements $a \in G$ of norm $|a| \leqslant x$ is finite for each real $x > 0$.

The most known prototype of an arithmetical semigroup is the multiplicative semigroup of all positive integers. Besides other expected examples of the multiplicative semigroups of all non-zero ideals in algebraic number fields, the category of all finite Abelian groups with the usual direct product operation and the norm function $|A| = \mathrm{card}(A)$ is one of the known non-standard examples.

Plainly, all theorems of the ordinary number theory depending only on the unique factorization property of positive integers remain valid (after a suitable rewriting if necessary) also for arithmetical semigroups. This concerns, in particular, Möbius inversion formulae or some basic properties of the zeta-function

$$\zeta_G(s) = \sum_{n\in G} |n|^{-s} = \prod_{p\in P(G)} (1-|p|^{-s})^{-1}$$

which shall be used in the sequel. We refer the reader to [7] or [2] for more details.

We are especially interested in subsets of G which themselves can be considered as arithmetical semigroups with the induced norm. Such subsets will be called *arithmetical subsemigroups* of G. A subset H is an arithmetical subsemigroup of G if and only if H is a free subsemigroup of G. This gives the following simple characterization:

PROPOSITION. *H is an arithmetical subsemigroup of an arithmetical semigroup G if and only if H is generated by a set $P(H) = \{m_i\}$ of elements of G subject to the following condition*:

$m_{i_1} m_{i_2} \ldots m_{i_s} = m_{j_1} m_{j_2} \ldots m_{j_r}$ implies that m_{i_k} $(1 \leqslant k \leqslant s)$ is equal to some m_{j_t} $(1 \leqslant t \leqslant r)$, and conversely.

Thus in order that a set $\{m_i\}$ of elements of G be the set of generators of an arithmetical subsemigroup of G it is sufficient that the m_i's be coprime in pairs (within G). (This condition is not necessary even in the case of positive integers as the set $\{m_1 = 9, m_2 = 10, m_3 = 15\}$ shows.) The latter condition is certainly satisfied if $\{m_i\} \subset P(G)$ and, consequently, Nymann's divisible sets from [5] are very special instances of arithmetical subsemigroups of the set of positive integers. Therefore, all the results from §1 of [5] are immediate consequences of general results on arithmetical semigroups from [7].

Another type of arithmetical subsemigroups can be constructed as follows. Given an arithmetical subsemigroup H of G and a positive integer k, the set $\{n^k;\ n \in H\}$ is again an arithmetical subsemigroup of G. Thus the set $\{n^k;\ n \in G\}$ is an arithmetical subsemigroup of G for every positive k.

2. The general case. Let G be an arithmetical semigroup and H an arithmetical subsemigroup of G. For $x_i \in G$ $(1 \leqslant i \leqslant k)$ we define the *H-greatest common divisor* of the x_i's by $(x_1, \ldots, x_k)_H = h$ if h is an element from H with the largest possible norm that divides each x_i $(1 \leqslant i \leqslant k)$.

So, for instance, the G-greatest common divisor is nothing else as the greatest common divisor in the usual sense. If $H = \{n^r; n \in G\}$, then the H-greatest common divisor extends the r-th power greatest common divisor introduced by Benkoski. Moreover, if $k = 1$ and $r > 1$, then $(x)_H = 1$ if and only if x is r-free.

Using the usual technique it can be shown that $(x_1, \ldots, x_k)_H$ always exists and is uniquely determined.

If k is a positive integer, then $T_k(H, x)$ will denote the number of k-tuples $[x_1, \ldots, x_k]$ of elements of G such that $|x_i| \leqslant x$ $(1 \leqslant i \leqslant k)$ and $(x_1, \ldots, x_k)_H = 1$.

Before stating the theorem we recall some concepts and notation from [7]. An arithmetical semigroup G is called *δ-regular* if its counting function $G(x)$ satisfies the condition

$$G(x) = x^\delta L(x),$$

where $L(x)$ is defined for all $x > 0$ and

$$\lim_{x\to\infty} \frac{L(ax)}{L(x)} = 1 \quad \text{for every } a > 0.$$

It can be proved (see [7]) that if G is an arithmetical semigroup such that $G(x)$ is δ-regular, then the series

$$\zeta_G(s) = \sum_{n \in G} |n|^{-s}$$

is convergent for every $s > \delta$ and divergent for every $s < \delta$.

Finally, we define (after Wegmann [7]) the following functions:

$$r(x, t) = \frac{t^\delta G(x/t)}{G(x)} - 1, \quad x > 0,\ t > 0,$$

$$R(x, \varepsilon) = \sup\{a;\ |r(x, y)| < \varepsilon,\ y \leqslant a\}, \quad x \geqslant 1, a > 0, \varepsilon > 0.$$

Then

$$\lim_{x\to\infty} r(x, a) = 0 \quad \text{for every } a > 1,$$

$$|r(x, y)| < \varepsilon \quad \text{for every } y < R(x, \varepsilon),$$

$$\lim_{x\to\infty} R(x, \varepsilon) = \infty \quad \text{for every } \varepsilon > 0.$$

THEOREM 1. *Let G be a δ-regular arithmetical semigroup such that the functions $r(x, t)$ are uniformly bounded for all x and t. Further, let k be a positive integer and H an arithmetical subsemigroup of G such that the series*

$$\sum_{h \in H} |h|^{-k\delta} = \zeta_H(k\delta) \tag{7}$$

converges. Then

$$T_k(H, x) = [\zeta_H^{-1}(k\delta)+o(1)][G(x)]^k.$$

Of course, if $k \geqslant 2$, then (7) as a subseries of an absolutely convergent series $\sum_{n\in G} |n|^{-k\delta}$ is always convergent, and therefore the condition imposed on H is actual only in the case $k = 1$. If $k = 1$, then (7) is convergent if and only if the series $\sum_{m\in P(H)} |m|^{-\delta}$ converges ([7], Satz 1.3).

Proof of Theorem 1. Since $(a_1, \dots, a_k)_H$ is determined for every k-tuple $[a_1, \dots, a_k]$ of elements of G, we have

$$[G(x)]^k = \sum_{\substack{|b|\leqslant x\\ b\in H}} \sum_{\substack{(a_1,\dots,a_k)_H=b\\ |a_i|\leqslant x,\ a_i\in G}} 1 = \sum_{\substack{|b|\leqslant x\\ b\in H}} \sum_{\substack{(c_1,\dots,c_k)_H=1\\ |c_i|\leqslant x/|b|,\ c_i\in G}} 1 = \sum_{\substack{|b|\leqslant x\\ b\in H}} T_k\left(H, \frac{x}{|b|}\right).$$

The Möbius inversion formula yields

$$T_k(H, x) = \sum_{\substack{|b|\leqslant x\\ b\in H}} \mu_H(b)\left[G\left(\frac{x}{|b|}\right)\right]^k, \tag{8}$$

where μ_H denotes the Möbius function associated with H.

To estimate the right-hand side of (8) we shall modify Wegmann's idea from the proof of Satz 2.5 in [7] as follows.

Since

$$G\left(\frac{x}{|b|}\right) = |b|^{-\delta}G(x)\big(1+r(x, |b|)\big),$$

we have

$$\frac{T_k(H, x)}{[G(x)]^k} = \sum_{\substack{|b|\leqslant x\\ b\in H}} \frac{\mu_H(b)}{|b|^{k\delta}} + \sum_{n=1}^{k}\binom{k}{n}\sum_{\substack{|b|\leqslant x\\ b\in H}} \frac{\mu_H(b)}{|b|^{k\delta}} r^n(x, |b|). \tag{9}$$

Inasmuch as (7) is convergent, to complete the proof it is sufficient to show that the last sum of (9) is arbitrarily small for sufficiently large x. But for every $\varepsilon > 0$ we have

$$\begin{aligned}\left|\sum_{\substack{|b|\leqslant x\\ b\in H}} \frac{\mu_H(b)}{|b|^{k\delta}} r^n(x, |b|)\right| &\leqslant \left|\sum_{\substack{|b|\leqslant R(x,\varepsilon)\\ b\in H}}\right| + \left|\sum_{\substack{R(x,\varepsilon)<|b|\leqslant x\\ b\in H}}\right| \\ &\leqslant \varepsilon \sum_{\substack{|b|\leqslant R(x,\varepsilon)\\ b\in H}} |b|^{-k\delta} + C \sum_{\substack{R(x,\varepsilon)<|b|\leqslant x\\ b\in H}} |n|^{-k\delta} \\ &\leqslant \varepsilon\zeta_H(k\delta) + C \sum_{|b|>R(x,\varepsilon)} |b|^{-k\delta}\end{aligned}$$

(C is the constant specified by the hypotheses of the theorem); this completes the proof.

From the point of view of number theory the most interesting δ-regular arithmetical semigroups are those satisfying Knopfmacher's Axiom A [2] which reads:

AXIOM A. *There exist positive constants A and δ and a constant η with $0 \leqslant \eta < \delta$ such that*

$$G(x) = Ax^{\delta} + O(x^{\eta}) \quad \text{as } x \to \infty.$$

The next theorem generalizes Knopfmacher's Theorem 4.5.12 of [2] in the spirit of our Theorem 1. Since both our and Knopfmacher's proofs are based on the same ideas, the reader can derive the subsequent estimates under the assumption of Axiom A also from our proof.

THEOREM 2. *Let G be an arithmetical semigroup satisfying Axiom A and let H be an arithmetical subsemigroup of G. If $k \geqslant 2$, then*

$$T_k(H, x) = A^k \zeta_H^{-1}(k\delta)x^{k\delta} + \begin{cases} O(x^{(k-1)\delta+\eta}) & \text{if } k > 2, \text{ or } k = 2 \text{ and } \eta > 0, \\ O(x^{\delta}\log x) & \text{if } k = 2,\ \eta = 0. \end{cases}$$

4. The case $H = G$. It seems that, assuming only the δ-regularity of $G(x)$, nothing interesting can be expected about the error estimate for $T_k(G, x)$. Adding new hypotheses to those of Theorem 1 we obtain

THEOREM 3. *Let G be a δ-regular arithmetical semigroup such that $|r(x, t)| \leqslant f(x)$ for every $t < x$ with $f(x) = o(1)$. Then*

$$T_k(G, x) = \left[\zeta_H^{-1}(k\delta) + O\left(f(x) + x^{-\delta(k-1)+\varepsilon}\right)\right][G(x)]^k$$

for every integer $k \geqslant 2$ and every $\varepsilon > 0$.

After consulting the proof of Satz 2.5 of [7], the proof of Theorem 3 can safely be left to the reader.

Let G be δ-regular and let $P \subset P(G)$. Then G_P will denote the arithmetical subsemigroup of G generated by $P(G) - P$. Wegmann proved ([7], Sätze 2.1 and 2.2) that

$$G_P(x) = \sum_{\substack{|a| \leqslant x \\ a \in G_P}} 1 = \left[\prod_{p \in P} (1 - |p|^{-\delta}) + o(1)\right] G(x) \tag{10}$$

provided one of the following conditions holds:

(i) P is finite and G is an arbitrary δ-regular arithmetical semigroup,

(ii) P is infinite with convergent $\sum_{p \in P} |p|^{-\delta}$ and G is δ-regular such that there exists a constant C with $|r(x, |p|)| < C$ for all x and $p \in P$, $|p| \leqslant x$.

This implies that in either case $G_P(x)$ is δ-regular together with $G(x)$. Thus for $H = G_P = G$ we get from Theorem 1 the following

COROLLARY 1. *Let G be given as in Theorem 1. If $P \subset P(G)$ is such that $\sum_{p\in P} |p|^{-\delta} < \infty$, then*

$$T_k(G_P, x) = \Big[\prod_{p\notin P}(1-|p|^{-k\delta})+o(1)\Big]\,[G_P(x)]^k$$

for every integer $k \geqslant 2$.

This is the result we announced above. This shows that (6) (i.e. Corollary 2 of [5]) remains true also if V is generated by a set of primes such that the series of reciprocal values of excluded primes converges. So, for instance, if $P(n)$ denotes the probability that two integers not exceeding n and having no prime divisor amongst the prime twins, chosen at random, are relatively coprime, then

$$\lim_{n\to\infty} P(n) = \prod_p{}^{*}(1-p^{-2}),$$

the asterisk * indicating that the product is over the primes which are not prime twins.

If the series $\sum_{p\notin P} |p|^{-\delta}$ is divergent, then G_P has zero asymptotic density within G. It seems that in this case the corresponding limit in (6) vanishes. In the next lines we show how an asymptotic formula with error estimate can be obtained for the original Nymann's result (6).

Suppose now that G satisfies Axiom A. Then in the case $H = G$ our Theorem 2 reduces to Theorem 4.5.12 of [2], which immediately implies (1). As mentioned above, Nymann's divisible sets are special instances of arithmetical subsemigroups of the semigroup of positive integers, and therefore also (5) is obtainable in turn from Theorem 2 (or Theorem 4.5.12).

Return again to (6). Nymann's proof of (6) is based on the observation that the set S of positive integers which are coprime to a given finite set of primes is divisible and $S(x) = Ax+O(1)$ for suitable A. But if G is an arithmetical semigroup satisfying Axiom A (what the set of positive integers is) and a is an element of G, then ([2], Theorem 4.1.3) the set of all elements of G that are coprime to a forms an arithmetical subsemigroup of G again satisfying Axiom A (a related result for δ-regular semigroups is given in (10)). Thus (6) can be derived from Theorem 2 even with an error estimate.

4. The case $H = \{n^r; n \in G\}$. If $r = 1$, then $H = G$, and therefore we can suppose that r is a positive integer greater than or equal to 2. But if $r \geqslant 2$ and $H = \{n^r; n \in G\}$, then the hypothesis regarding the convergence of (7) is again superfluous, and so the case $k = 1$ can be again incorporated in our considerations. Theorem 4 below is an analogue of Theorem 3.

In this section we return to the Benkoski shorter notation. Namely, $T_k^r(x)$ will denote the number $T_k(H, x)$ provided $H = \{n^r; n \in G\}$.

THEOREM 4. *Let G be given as in Theorem 3. If $k \geqslant 1$ and $r \geqslant 2$ are integers, then for every $\varepsilon > 0$ we have*

$$T_k^r(x) = \left[\zeta_G^{-1}(rk\,\delta) + O\left(f(x) + x^{-\delta(k-1)/r+\varepsilon}\right)\right][G(x)]^k.$$

The proof runs again along modified Wegmann's ideas of the proof of Satz 2.5 in [7] if we take into account that $\zeta_H(k\delta) = \zeta_G(rk\,\delta)$ and $\mu_H(b^r) = \mu_G(b)$ hold in this case.

Further, we have

THEOREM 5. *Let G be an arithmetical semigroup satisfying Axiom A. If $k \geqslant 1$ and $r \geqslant 2$ are two integers, then*

$$T_k^r(x) = A^k \zeta_G^{-1}(rk\,\delta)x^{k\delta} + \begin{cases} O(x^{(k-1)\delta+\eta}) & \text{if } k > 2, \text{ or } k = 2 \text{ and } \eta > 0, \\ O(x^{\delta}\log x) & \text{if } k = 2,\ \eta = 0, \\ O(x^{\delta/r}) & \text{if } k = 1,\ \eta < \delta/r, \\ O(x^{\delta/r}\log x) & \text{if } k = 1,\ \eta = \delta/r, \\ O(x^{\eta}) & \text{if } k = 1,\ \eta > \delta/r. \end{cases}$$

We leave the tedious proof to the reader, noticing only that (8) reduces to

$$T_k^r(x) = \sum_{\substack{|b| < x^{1/r} \\ b \in G}} \mu_G(b)\left[A(x/|b|^r)^{\delta} + O\left((x/|b|^r)^{\eta}\right)\right]^k.$$

The discussion of relations to Benkoski's results is dispensable.

5. Related cases. Theorems 1 and 4 reduce to Wegmann's Satz 2.5 about the density of r-free elements in G provided $k = 1$. However, the result on the density of r-free elements of G is a particular case of the next corollary to Theorem 1. Beforehand a definition:

Given an arithmetical subsemigroup H of G, an element n of G will be called *H-free* if the identity element 1 is the only divisor of n belonging to H.

COROLLARY 2. *Under the hypotheses of Theorem 1 the density of H-free elements of a δ-regular arithmetical semigroup is equal to $\zeta_H^{-1}(\delta)$.*

The next corollary contains also some of already discussed results.

COROLLARY 3. *Let G be given as in Theorem 1. Let $S = \{p_\lambda\}_{\lambda\in\Lambda}$ be a set of distinct primes of G and $\{a_\lambda\}_{\lambda\in\Lambda}$ a set of positive integers. If H is generated by $P(H) = \{p_\lambda^{a_\lambda}\}_{\lambda\in\Lambda}$, then for every integer $k \geqslant 2$ we have*

$$T_k(H, x) = \left[\prod_{\lambda\in\Lambda}(1 - |p|^{-a_\lambda k\delta}) + o(1)\right][G(x)]^k.$$

Result (4) of Benkoski gives rise to the following question:

Given an arithmetical semigroup G and two its arithmetical subsemigroups H and S, what can be said about the number $T_k(H, S, x)$ of

k-tuples $[x_1, \ldots, x_k]$ of elements of G with $|x_i| \leqslant x$ and $(x_1, \ldots, x_k)_H \in S$? **(P 1231)**

The following corollary gives a partial answer to this problem.

COROLLARY 4. *Let G and H be given as in Theorem* 1. *Assume that there exist two arithmetical subsemigroups S_1 and S_2 of H such that every element h of H can be uniquely represented in the form $h = s_1 s_2$ with $s_i \in S_i$ $(i = 1, 2)$. Then*

$$T_k(H, S_1, x) = [\zeta_{S_2}^{-1}(k\delta) + o(1)][G(x)]^k.$$

The corollary follows immediately from the fact that $T_k(H, S_1, x) = T_k(S_2, x)$ in this case.

In particular, putting $P \subset P(G)$, $H = \{n^r; n \in G\}$, $S_1 = \{n^r;\ n \in G_{P(G)-P}\}$ and $S_2 = \{n^r; n \in G_P\}$ in Corollary 4 we see that Benkoski's "finiteness" assumptions in (4) are redundant.

REFERENCES

[1] S. J. Benkoski, *The probability that k integers are relatively r-prime*, Journal of Number Theory 8 (1976), p. 218-223.

[2] J. Knopfmacher, *Abstract analytic number theory*, Amsterdam - New York 1975.

[3] D. N. Lehmer, *An asymptotic evaluation of certain totient sums*, American Journal of Mathematics 22 (1900), p. 293-335.

[4] J. E. Nymann, *On the probability that k positive integers are relatively prime*, Journal of Number Theory 4 (1972), p. 469-473.

[5] — *On the probability that k positive integers are relatively prime, II*, ibidem 7 (1975), p. 406-412.

[6] — and W. J. Leahey, *On the probability that integers chosen according to the binomial distribution are relatively prime*, Acta Arithmetica 31 (1976), p. 205-211.

[7] H. Wegmann, *Beiträge zur Zahlentheorie auf freien Halbgruppen, I*, Journal für die reine und angewandte Mathematik 221 (1966), p. 20-43.

MATHEMATICAL INSTITUTE
SLOVAK ACADEMY OF SCIENCES
BRATISLAVA

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