COLLOQUIUM MATHEMATICUM

VOL. XV 1966 FASC. 2

ON MYCIELSKI'S PROBLEM ON SYSTEMS OF ARITHMETICAL PROGRESSIONS

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We shall say that a system of k arithmetical progressions

$$\ldots, a_i-2n_i, a_i-n_i, a_i, a_i+n_i, a_i+2n_i, \ldots \tag{1}$$

$(i=1,2,\ldots,k)$ has the property F if every integer belongs to exactly one of these progressions.

J. Mycielski made the following conjecture:

If a system (1) of progressions has the property F and if in one of these progressions

$$n_{i_0} = \prod_{t=1}^{r} p_t^{\lambda_t},$$

where p_t are different primes, then

$$k \geqslant 1+\sum_{t=1}^{r}\lambda_t(p_t-1).$$

The validity of this conjecture follows from

THEOREM I. *If* (1) *has the property* F, *then all numbers*

$$a_{i_0}+c_t q_t p_t^{a_t} \tag{2}$$

(*where* $t=1,2,\ldots,r$; $c_t=1,2,\ldots,p_t-1$; $q_t=n_{i_0}/p_t^{\lambda_t}$; $a_t=0,1,\ldots,\lambda_t-1$) *belong to distinct progressions of* (1).

Note. The number of numbers in (2) is equal to $\sum_{t=1}^{r}\lambda_t(p_t-1)$; a_{i_0} cannot fall in the same progression as any of numbers in (2) and the conjecture follows.

To prove our theorem we shall need the following

LEMMA. *If* (1) *has the property* F *and*

$$a_{i_0}+c_t q_t p_t^{a_t} = a_j+hn_j, \tag{3}$$

then n_j can be written in the form

$$n_j = Kp_t^{\beta_t},$$

where $\beta_t > a_t$ and $p_t \nmid K$.

Proof. If $j = i_0$, then the validity of the lemma follows from inequality $a_t < \lambda_t$.

Suppose then $j \neq i_0$. Putting

$$(q_t p_t^{a_t}, n_j) = d \tag{4}$$

we can write $n_j = Ld$. From (3) it follows that $a_{i_0} - a_j = Md$ (M and L are integers). Now consider the diophantine equation

$$M = h_1 L - h_2 \frac{n_{i_0}}{d}.$$

If $(L, n_{i_0}/d) = 1$, then this equation is solvable in integers, and hence the equation

$$a_{i_0} - a_j = h_1 n_j - h_2 n_{i_0}$$

is solvable in integers too. This is a contradiction, because every integer belongs to exactly one of progressions (1). Hence $(L, n_{i_0}/d) > 1$. From (4) it follows that $(q_t p_t^{a_t}/d, L) = 1$ and so we have

$$\left(\frac{n_{i_0}}{d}; L\right) = \left(\frac{q_t p_t^{a_t}}{d} p_t^{\lambda_t - a_t}, L\right) = (p_t^{\lambda_t - a_t}, L) > 1.$$

This implies $p_t \mid L$, whence and from (4) we infer that $\beta_t > a_t$. Hence the Lemma is proved.

Proof of Theorem I. We will show that two distinct numbers of the form (2) cannot belong to the same progression. We must distinguish two cases. The first case is

$$a_{i_0} + c_t q_t p_t^{a_t} = a_j + h n_j, \tag{5}$$

$$a_{i_0} + c_t' q_t p_t^{a_t'} = a_j + h' n_j. \tag{6}$$

Let $a_t' \geqslant a_t$ (the case $a_t' \leqslant a_t$ is dual). By a subtraction (5) from (6) we get

$$q_t p_t^{a_t}(c_t' p_t^{a_t' - a_t} - c_t) = h'' n_j \tag{7}$$

(where $h'' = h' - h$). In view of (5) the lemma implies that $n_j = Kp_t^{\beta_t}$ with $\beta_t > a_t$. Let us divide (7) by $p_t^{a_t}$; we shall have then

$$q_t c_t' p_t^{a_t' - a_t} - q_t c_t = h'' K p_t^{\beta_t - a_t}.$$

But this is a contradiction since the right-hand side is divisible by p_t and the left-hand side is not (if $a_t' = a_t$, then $c_t' \neq c_t$, because numbers in (5) and (6) are distinct).

The second case is

$$a_{i_0}+c_t q_t p_t^{a_t} = a_j+hn_j, \tag{8}$$

$$a_{i_0}+c_m q_m p_m^{a_m} = a_j+h'n_j, \tag{9}$$

where $t \neq m$. Let us subtract (9) from (8):

$$c_t q_t p_t^{a_t} - c_m q_m p_m^{a_m} = h''n_j. \tag{10}$$

By (9) and the Lemma we have $n_j = Kp_m^{\beta_m}$ with $\beta_m > a_m$.
Dividing (10) by $p_m^{a_m}$ we have

$$c_t p_t^{a_t} \frac{q_t}{p_m^{a_m}} - c_m q_m = h''Kp_m^{\beta_m - a_m}.$$

Now $c_m q_m$ is not divisible by p_m, but the remaining expressions in the last equality are; this is a contradiction and the proof of Theorem I is finished.

The bound

$$1+\sum_{t=1}^{r} \lambda_t(p_t-1)$$

is the best possible. Namely, we can show

THEOREM II. *Let*

$$\ldots, a-2n, a-n, a, a+n, a+2n, \ldots \tag{11}$$

be an arbitrary arithmetical progression. If

$$n = \prod_{t=1}^{r} p_t^{\lambda_t},$$

where p_t are different primes, then the system that contains (11) *and all progressions of the form*

$$\ldots, (a+c_t u_{a_t})-2p_t u_{a_t}, (a+c_t u_{a_t})-p_t u_{a_t}, (a+c_t u_{a_t}), \tag{12}$$
$$(a+c_t u_{a_t})+p_t u_{a_t}, (a+c_t u_{a_t})+2p_t u_{a_t}, \ldots$$

(*where* $t=1,2,\ldots,r$; $c_t = 1,2,\ldots,p_t-1$; $a_t = 0,1,\ldots,\lambda_t-1$; $u_{a_t} = p_1^{\lambda_1}p_2^{\lambda_2}\ldots p_{t-1}^{\lambda_{t-1}}p_t^{a_t}$) *has the property* F.

Proof. Since all moduli $p_t u_{a_t}$ divide n, it is sufficient to show that every number between a and $a+n$ belongs to exactly one progression.

We might easily show by methods used in the proof of Theorem I that no integer belongs to two different progressions.

Let the number $b = a+c$ lie between a and $a+n$. Suppose that $u_{at} \mid c$, but $p_t u_{a_t} \nmid c$. Then c can be written in the form $c = du_{a_t}p_t+e$

(where $d \geqslant 0$, $0 < e < p_t u_{a_t}$). We have $e = f u_{a_t}$, where f is one of the numbers $1, 2, \ldots, p_t - 1$; hence b belongs to the progression

$$\ldots, (a+fu_{a_t})-2p_t u_{a_t}, (a+fu_{a_t})-p_t u_{a_t}, (a+fu_{a_t}),$$
$$(a+fu_{a_t})+p_t u_{a_t}, (a+fu_{a_t})+2p_t u_{a_t}, \ldots$$

which is included in (12). This concludes the proof of Theorem II.

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Reçu par la Rédaction le 22. 4. 1965