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A PROOF OF PH. HALL'S THEOREM ON DIMENSION SUBGROUPS

BY

A. GRZEGOREK (WROCŁAW)

Let G be a group and let QG be its group ring over rationals. If $A = \{\sum a_i g_i \in QG: \sum a_i = 0\}$ is the augmentation ideal in QG, then $(1+A^n)\cap G = D_n$ is a subgroup of G, called the *n-th dimension subgroup.* If $\{G_n\}$ is the lower central series of G, then it is not difficult to show that $G_n \subset D_n$. In 1957 Ph. Hall proved that D_n/G_n is periodic, in other words — D_n is the isolator of G_n. The proof of Ph. Hall's theorem was published in [3], which is not easily available. Here we present another, perhaps somewhat simpler proof of this very important theorem.

The proof reduces easily to the case where G is a finitely generated, torsion-free nilpotent group. Therefore, we assume that G is such a group and we denote by c its nilpotency class.

For $g, h \in G$ we write

$$(g, h) = g^{-1}h^{-1}gh.$$

If A and B are two subgroups of G, then we write

$$(A, B) = \mathrm{gp}\{(a, b): a \in A,\ b \in B\}.$$

Let $G = G_1 \supset G_2 \supset \ldots \supset G_{c+1} = \{e\}$ be the lower central series of G, i.e., $G_n = (G_{n-1}, G)$, $2 \leqslant n \leqslant c+1$. Put

$$F_n = \{g \in G: \exists k\ g^k \in G_n\}, \quad 1 \leqslant n \leqslant c+1.$$

THEOREM 1. *The sets F_n, $1 \leqslant n \leqslant c+1$, form a central series for G. Moreover, $(F_n, F_m) \subset F_{n+m}$.*

Proof. It is easy to verify that if $x, y, z \in G$, then

$$(1)\qquad (xy, z) = (x, z)(y, z)((x, z), y)(((x, z), y), (y, z)).$$

Let $C_r(g, h)$ be an r-fold commutator in g and h and let $k > 0$ be an integer. Applying (1) we obtain (by induction on r and k)

$$(2)\qquad C_r(g, h)^{k^r} = C_r(g^k, h^k) \prod_{i=r+1}^{c} \tau_i(g, h),$$

where each $\tau_i(g, h)$ is a product of i-fold commutators in g and h.

We have also

$$(3) \qquad (gh)^k = g^k h^k \prod_{j=2}^{c} \tau'_j(g, h),$$

where each $\tau'_j(g, h)$ is a product of j-fold commutators in g and h.

It is known (see [1], Lemma 2.8) that if H is a subgroup of a finitely generated nilpotent group G, then the set $\{g \in G\colon \exists\, k\ g^k \in H\}$ is equal to the isolator of H, hence it is also a subgroup of G. This follows also from (2) and (3), namely, one can prove that if $x^k, y^k \in H$, then $(xy)^{kk^2\ldots k^c} \in H$. Thus, for every $n = 1, 2, \ldots, c$, F_n is a subgroup of G.

Now we prove the inclusion $(F_n, F_m) \subset F_{n+m}$ which will imply that the series $\{F_n\}$ is central. For this, assume that $g^k \in G_n$ and $h^k \in G_m$. By (2),

$$(g, h)^{k^2} = (g^k, h^k)\tau_3(g, h)\prod_{i=4}^{c} \tau_i(g, h).$$

Observe that $(g^k, h^k) \in G_{n+m}$. Using (2) and (3) we obtain

$$(g, h)^{k^2k^3} = (g^k, h^k)^{k^3}\tau_3(g^k, h^k)\prod_{i=4}^{c} \tau''_i(g, h),$$

where τ''_i is a product of i-fold commutators in g and h.

Continuing this process we get

$$(g, h)^{k^2k^3\ldots k^{n+m-1}} \in G_{n+m},$$

which completes the proof of the theorem.

Remark 1. It follows immediately from the definition of F_n that F_n/F_{n+1} is either torsion-free or trivial. But the identity $F_n = F_{n+1}$ is impossible, since the length of every central series of G must be greater than or equal to c. Thus, for every $n = 1, 2, \ldots, c$, the quotient F_n/F_{n+1} is torsion-free.

Remark 2. F_n is finitely generated as a subgroup of a finitely generated nilpotent group. Hence F_n/F_{n+1} is finitely generated.

Let QG be the group algebra of G over Q and let A be the augmentation ideal, i.e.,

$$A = \Big\{\sum a_i g_i \in QG\colon \sum a_i = 0\Big\}.$$

Let A^n be the two-sided ideal of QG generated by all products $a_1 a_2 \ldots a_n$, where $a_i \in A$, $1 \leqslant i \leqslant n$. Denote by D_n the n-th dimension subgroup of G over Q, i.e.,

$$D_n = (1+A^n)\cap G.$$

Jennings [4] has proved that the dimension subgroups form a central series and that each quotient D_n/D_{n+1}, $1 \leqslant n \leqslant c$, is torsion-free. For more details concerning those subgroups see [2].

Our aim is to prove

THEOREM 2. *For every n, $1 \leqslant n \leqslant c+1$, we have $F_n = D_n$.*

First we prove the following

LEMMA 1. *If $\{E_n\}$ is a central series of G and, for every n, E_n/E_{n+1} is torsion-free, then each subgroup E_n is isolated.*

Proof. Assume that $g \in G$ and $g^k \in E_n$. Let $M = \max\{m: g \in E_m\}$. If $M < n$, then $g^k \in E_{M+1}$. As $g \in E_M$ and E_M/E_{M+1} is torsion-free, we obtain $g \in E_{M+1}$, contrary to the definition of M.

COROLLARY. *For every n, $1 \leqslant n \leqslant c$, $F_n \subset D_n$.*

Proof. Since $\{D_n\}$ is a central series, we have $G_n \subset D_n$. By Lemma 1, D_n is isolated. Thus $F_n \subset D_n$ as F_n is the smallest isolated subgroup containing G_n.

We select a sequence $g_1, g_2, \ldots, g_d$ of elements of G such that if $A_i = \mathrm{gp}\{g_i, g_{i+1}, \ldots, g_d\}$, then

(i) $G = A_1 \supset A_2 \supset \ldots \supset A_d \supset \{e\}$ is a refinement of the central series $\{F_n\}$;

(ii) A_i/A_{i+1} is an infinite cyclic group.

This is done as follows. By Remarks 1 and 2, F_n/F_{n+1} is a torsion-free, finitely generated Abelian group. Hence it is a finite direct product of infinite cyclic groups, say, $(g_1^{(n)}), \ldots, (g_{k_n}^{(n)})$. We take in G the coimages of the elements

$$g_1^{(1)}, \ldots, g_{k_1}^{(1)}, \ldots, g_1^{(c)}, \ldots, g_{k_c}^{(c)}$$

and we denote them by $g_1, g_2, \ldots, g_d$. We call this sequence a *basis* of G.

Observe that every element $g \in G$ has a unique representation in the form

$$g = g_1^{a_1} g_2^{a_2} \ldots g_d^{a_d}, \qquad \text{where } a_1, a_2, \ldots, a_d \in Z.$$

In view of Lemma 4.1 in [1] the products

$$(4) \qquad (g_{i_1}^{\pm 1}-1)^{a_1} \ldots (g_{i_j}^{\pm 1}-1)^{a_j}, \qquad 1 \leqslant i_1 < \ldots < i_j \leqslant d,\ a_n > 0,$$

constitute a basis for the vector space A over Q.

For $g \in G$ let

$$w(g) = \max\{n: g \in F_n\}.$$

We define a straight product as

$$P_{\alpha,\beta} = \prod_{i=1}^{d} (g_i-1)^{\alpha_i}(g_i^{-1}-1)^{\beta_i},$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_d)$ and $\alpha_i, \beta_i \geqslant 0$ for $i = 1, 2, \ldots, d$.

Let

$$W(P_{\alpha,\beta}) = \sum_{i=1}^{d} w(g_i)(\alpha_i + \beta_i)$$

and

$$W\Big(\sum_{\alpha,\beta} c_{\alpha,\beta} P_{\alpha,\beta}\Big) = \min\{W(P_{\alpha,\beta}) \colon c_{\alpha,\beta} \neq 0\}.$$

Now define the weight of an element $a \in A$ by

$$W(a) = \max\Big\{W\Big(\sum c_{\alpha,\beta} P_{\alpha,\beta}\Big) \colon a = \sum c_{\alpha,\beta} P_{\alpha,\beta}\Big\}.$$

LEMMA 2. *$W(a) \geqslant n$ if and only if $a \in A^n$.*

Proof. The necessity is an immediate consequence of the definition of $W(a)$ and of the inclusion $F_i - 1 \subset A^i$ (see the Corollary). To prove the sufficiency take the product $(h_1 - 1) \ldots (h_k - 1)$, where each $h_i \in G$ and $k \geqslant n$, and replace each factor $h_i - 1$ by its presentation as a linear combination of basic elements (4). Hence

$$(h_1 - 1) \ldots (h_k - 1) = \sum_{\alpha} c_\alpha (g_{a_1}^{\pm 1} - 1) \ldots (g_{a_m}^{\pm 1} - 1), \qquad m \geqslant k \geqslant n.$$

If $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_m$, we have a straight product of weight less than or equal to n. Otherwise, applying the identity

$$\begin{aligned}(y-1)(x-1) = (x-1)(y-1) + (x-1)(y-1)\big((y, x) - 1\big) + \\ + (x-1)\big((y, x) - 1\big) + (y-1)\big((y, x) - 1\big) + \big((y, x) - 1\big)\end{aligned}$$

a sufficient number of times, we obtain a straight presentation (i.e., an expression being the linear combination of straight products) of the product $(g_{a_1}^{\pm 1} - 1) \ldots (g_{a_m}^{\pm 1} - 1)$. Observe that its weight is less than or equal to n, since for every $x, y \in G$ we have

$$w\big((y, x)\big) \geqslant w(y) + w(x).$$

This completes the proof of Lemma 2.

Proof of Theorem 2. It remains to prove that $D_n \subset F_n$.

Let $g - 1 \in A^n$ and let $g = g_i^{\alpha_i} \ldots g_d^{\alpha_d}$, where $\alpha_i \neq 0$. We have to show that $w(g_i) \geqslant n$. It suffices to prove that every straight presentation of $g - 1$ includes either $c(g_i - 1)$ or $c(g_i^{-1} - 1)$, where $c \neq 0$ is a rational. Indeed, in virtue of Lemma 2 we have $W(g-1) \geqslant n$, whence $W(g_i^{\pm 1} - 1) \geqslant n$, which implies $w(g_i) \geqslant n$.

Assume that there exists a straight presentation which does not include the term $c(g_i^{\pm 1} - 1)$.

The equality

$$(5)\qquad (x-1)(x^{-1}-1) = -(x-1)-(x^{-1}-1)$$

enables us to pass from a straight product to the sum of basic products of form (4).

Note that g_i-1 or $g_i^{-1}-1$ can be obtained only from the terms $c(g_i-1)^k(g_i^{-1}-1)^l$ provided $k > 0$ and $l > 0$. Applying (5) to this product we get

$$\begin{aligned}(g_i-1)^k(g_i^{-1}-1)^l &= (g_i-1)^{k-1}[-(g_i-1)-(g_i^{-1}-1)](g_i^{-1}-1)^{l-1}\\ &= -(g_i-1)^k(g_i^{-1}-1)^{l-1}-(g_i-1)^{k-1}(g_i^{-1}-1)^l\end{aligned}$$

and, finally, by induction on k and l,

$$(6)\qquad (g_i-1)^k(g_i^{-1}-1)^l = c[-(g_i-1)-(g_i^{-1}-1)]+R,$$

where R is a linear combination of $(g_i^{\pm 1}-1)^m$ with $m > 1$.

Using the formula

$$xy-1 = (x-1)(y-1)+(x-1)+(y-1)$$

we obtain the basic presentation of $g-1$:

$$g-1 = |a_i|(g_i^{\pm 1}-1)+\ldots+|a_d|(g_d^{\pm 1}-1)+\ldots$$

Since this presentation includes either $a_i(g_i-1)$ (if $a_i > 0$) or $|a_i|(g_i^{-1}-1)$ (if $a_i < 0$), equality (6) gives us a contradiction.

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