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CONNECTEDNESS OF COMPLETE METRIC GROUPS

BY

T. CHRISTINE STEVENS (CAMBRIDGE)

1. Introduction. Topologies for subgroups of the real numbers that are weaker or stronger than the usual topology have been studied in [1], [3]–[8], [10], not only for their intrinsic interest but also as sources of useful examples and counterexamples. In the present paper we use a set $\{p_i: i \in \mathbf{Z}\}$ of positive real numbers to define norms $\| \ \|$ for the group S of dyadic rationals that are stronger than the usual norm and then examine the completions of S with respect to these norms. Our results provide a negative answer to a question posed by S. Mazur in *The Scottish Book* ([2], Problem 160, pp. 236–238): Must a complete metric group which is generated by every neighbourhood of the identity be connected? Indeed, if C is the completion of $(S, \| \ \|)$ and $f: C \to \mathbf{R}$ is the natural map, then C is always generated by every neighbourhood of the identity and f is always injective. The homomorphism f is surjective, however, if and only if $\sum_1^{\infty} p_i < \infty$, and thus C is "often" totally disconnected. To determine whether a given real number x is in $f(C)$, we need only ascertain whether a certain subseries of $\sum_1^{\infty} p_i$, derived from the binary expansion of x, converges.

Our principal theorems are precisely stated in Section 2 and proved in that section and in Section 4, while Section 3 contains preliminary results that establish important relationships among binary expansions, the value of the norm $\| \ \|$, and the image $f(C)$. In the final section we place our results in the context of the differing definitions of "connected" that were proposed by Cantor and Hausdorff.

2. Main results. We begin by establishing the notation that will be used throughout this paper. $\mathbf{R}$, $\mathbf{Q}$, $\mathbf{Z}$, N, and $| \ |$ denote, respectively, the real numbers, the rational numbers, the integers, the natural numbers, and the usual absolute value, and S is the set if dyadic integers (that is, the set of all real numbers of the form $b/2^n$, where $b, n \in \mathbf{Z}$). Since we will often be concerned with finite sums and infinite series, it is convenient to stipulate that the symbol $\sum x_i$ always denotes a finite sum; if the initial and final values of the index i are of interest, we will write this sum as $\sum_m^n x_i$. An infinite series will be written as $\sum_m^\infty x_i$. The symbol ■ marks the end of a proof.

We may now define the norms which form the subject of this paper. Let $\{p_i\colon i \in \boldsymbol{Z}\}$ be a set of positive real numbers such that $p_i \leqslant p_{i-1} \leqslant 2p_i$ for all $i \in \boldsymbol{Z}$ and $p_i \to 0$ as $i \to +\infty$. Our strategy is to construct the largest norm $\|\ \|$ on S such that $\|2^{-i}\| \leqslant p_i$ for all $i \in \boldsymbol{Z}$, and thus we define $\|\ \|$ by

$$(1) \qquad \|s\| = \inf\{\sum |a_i| p_i\colon s = \sum a_i 2^{-i},\ a_i \in \boldsymbol{Z}\} \quad \text{for} \quad s \in S.$$

Our principal results are contained in the following two theorems.

THEOREM 2.1. *Let $\{p_i\colon i \in \boldsymbol{Z}\}$ and $\|\ \|$ be as above. Then*

(i) *$\|\ \|$ is a norm on S and is stronger than $|\ |$;*

(ii) *if $(C, \|\ \|)$ is the completion of $(S, \|\ \|)$, then C is generated by every neighbourhood of the identity;*

(iii) *if $f\colon C \to \boldsymbol{R}$ is the uniformly continuous extension of the identity homomorphism $(S, \|\ \|) \to (S, |\ |)$, then f is injective.*

THEOREM 2.2. *Let $(C, \|\ \|)$ be as in Theorem 2.1. Then the following statements are equivalent:*

(i) *$\sum_1^\infty p_i$ does not converge in $(\boldsymbol{R}, |\ |)$.*

(ii) *$\boldsymbol{Q} \cap f(C) = S$.*

(iii) *$f(C) \neq \boldsymbol{R}$.*

(iv) *C is totally disconnected.*

(v) *$\|\ \|$ is strictly stronger than $|\ |$ on S.*

Before proceeding further, we observe that sets $\{p_i\colon i \in \boldsymbol{Z}\}$ satisfying the hypotheses of Theorem 2.1 are easy to find. We can, for example, let $p_i = 1$ if $i \leqslant 0$ and $p_i = 1/i$ if $i \geqslant 1$, or let $p_i' = 1/9$ if $i \leqslant 2$ and $p_i' = 1/i^2$ if $i \geqslant 3$. Then $\{p_i\}$ satisfies Theorem 2.2 (i), but $\{p_i'\}$ does not.

Although the proofs of 2.1 (iii) and 2.2 must await the discussion of binary expansions in Section 3, we can proceed without delay to the proofs of parts (i) and (ii) of 2.1. We begin with a useful observation which follows easily from the hypothesis that $p_{i-1} \leqslant 2p_i$ for all $i \in \boldsymbol{Z}$.

LEMMA 2.3. *For every $s \in S$,*

$$\|s\| = \inf\{\sum |a_i| p_i\colon s = \sum a_i 2^{-i},\ a_i \in \{0, \pm 1\}\}.$$

We now prove 2.1 (i). The triangle inequality and the fact that $\|s\| = \|-s\|$ for every $s \in S$ follow from (1). To prove the remainder of (i), it will suffice to show that $|s| < 1/2^j$ whenever $\|s\| < p_j$. If $\|s\| < p_j$, then Lemma 2.3

implies that s can be written in the form $\sum_m^n a_i 2^{-i}$, where $a_i \in \{0, \pm 1\}$ and $\sum_m^n |a_i| p_i < p_j$. Since $p_i \leqslant p_{i-1}$ for all $i \in \boldsymbol{Z}$, we conclude that $a_i = 0$ if $i \leqslant j$, whence $|s| \leqslant \sum_{j+1}^n |a_i| 2^{-i} < 1/2^j$.

To prove Theorem 2.1 (ii), let $\varepsilon > 0$ be given and let $B = \{c \in C: \|c\| < \varepsilon\}$. Since $\|2^{-i}\| \leqslant p_i$ and $p_i \to 0$ as $i \to +\infty$, B contains all 2^{-i} with i sufficiently large and thus generates a subgroup that contains S. Since $(S, \| \ \|)$ is dense in C, this subgroup also contains C. ■

3. Binary expansions. In this section we show how to use the binary expansion for an element $x \geqslant 0$ of $\boldsymbol{R}$ to determine whether $x \in f(C)$. If we agree always to use the terminating expansion for an element of S, then every positive real number x can be written uniquely in the form

$$x = \sum_{j=1}^{n} \sum_{i=i_j}^{k_j} 2^{-i}, \tag{2}$$

where i_j and k_j are integers such that $i_j \leqslant k_j \leqslant i_{j+1} - 2$ for all j; n is finite if $x \in S$, and otherwise $n = \infty$. This representation, which divides the binary expansion of x into blocks of ones separated by at least one zero, will be called the *standard representation of* x; the standard representation of 0 is simply the expansion in which all coefficients are zero.

If $x, y \in [0, +\infty)$ and $m \in \boldsymbol{Z}$, we will say that x and y are *m-equivalent* if, for every $i \leqslant m$, the coefficient of 2^{-i} in the standard representation of x equals the corresponding coefficient in the standard representation of y. In both this section and the next, we will have occasion to use the following fact about m-equivalence, the elementary proof of which we omit.

LEMMA 3.1. *Let $x > 0$ be in $\boldsymbol{R} \backslash S$, and let $\{x_m: m \in N\}$ be a sequence in $\boldsymbol{R}$ such that $|x_m - x| \to 0$. Then there is a subsequence $\{x_{m(j)}: j \in N\}$ of $\{x_m\}$ such that $x_{m(j)}$ and x are j-equivalent for all $j \in N$. The same conclusion holds if $\{x_m\}$ is a sequence of non-negative real numbers that converges to $x = 0$ in $(\boldsymbol{R}, | \ |)$.*

We now introduce the function D which will serve as the principal tool in our investigation of $(C, \| \ \|)$. For each $x \in [0, +\infty)$, D provides a weighted count of the frequency of changes from 0 to 1 in the standard representation of x. We define D as follows: Let $D(0) = 0$. If x is a positive real number with standard representation (2), then

$$D(x) = \begin{cases} \sum_{j=1}^{n} p_{i_j} & \text{if} \quad x \in S; \\ \sum_{j=1}^{\infty} p_{i_j} & \text{if} \quad x \in \boldsymbol{R} \backslash S \text{ and the series converges;} \\ \infty & \text{if} \quad x \in \boldsymbol{R} \backslash S \text{ and } \sum_{j=1}^{\infty} p_{i_j} \text{ diverges.} \end{cases}$$

The chief goal of this section is to prove the following proposition.

PROPOSITION 3.2. *Let x be a non-negative real number. Then $x \in f(C)$ if and only if $D(x) < \infty$.*

Our proof begins with an elementary lemma that compares the standard representation of an element s of S with other representations of s.

LEMMA 3.3. *Let $s > 0$ be an element of S with standard representation as in (2), and let $B = \sum_{i_1}^{k_1} 2^{-i}$. Suppose that s can also be written in the form $\sum_{m}^{p} a_i 2^{-i}$, where $a_i \in \{0, \pm 1\}$ and $a_m \neq 0$. Then $m \leqslant i_1$, and either*

(i) $\sum_{m}^{k_1} a_i 2^{-i} = B$

or

(ii) $a_{k_1+1} = -1$ *and* $\sum_{m}^{k_1+1} a_i 2^{-i} = B + 2^{-(k_1+1)}$.

Proof. To simplify the notation we let $\alpha = i_1$ and $\beta = k_1$. That $m \leqslant \alpha$ follows immediately from the inequality

$$2^{-\alpha} \leqslant s \leqslant \sum_{m}^{p} 2^{-i} < 2^{-(m-1)}$$

and the fact that $m, \alpha \in \mathbf{Z}$. Turning to the remainder of the lemma, we first observe that the inequalities

$$\sum_{\alpha}^{\beta} 2^{-i} \leqslant s < \sum_{\alpha}^{\beta+1} 2^{-i} \tag{3}$$

and $\left|\sum_{\beta+1}^{p} a_i 2^{-i}\right| < 2^{-\beta}$ together imply that

$$\sum_{\alpha}^{\beta-1} 2^{-i} < \sum_{m}^{\beta} a_i 2^{-i} < 2^{-(\alpha-1)} + 2^{-(\beta+1)}.$$

Therefore $\sum_{m}^{\beta} a_i 2^{-i}$, as an element of the subgroup of S that is generated by $2^{-\beta}$, must equal either $\sum_{\alpha}^{\beta} 2^{-i}$ or $2^{-(\alpha-1)}$. That is, either (i) occurs or $\sum_{m}^{\beta} a_i 2^{-i} = 2^{-(\alpha-1)}$, and to finish the proof it suffices to show that in the latter case $a_{\beta+1} = -1$. We accomplish this by noting that the inequality

$$s \geqslant 2^{-(\alpha-1)} + \sum_{\beta+2}^{p} a_i 2^{-i} > 2^{-(\alpha-1)} - 2^{-(\beta+1)} = \sum_{\alpha}^{\beta+1} 2^{-i}$$

not only holds when $a_{\beta+1} \geqslant 0$ but also contradicts (3). ∎

We now prove an important relationship between $\|s\|$ and $D(\mathring{s})$.

LEMMA 3.4. *For every non-negative* $s \in S$, $D(s) \leqslant \|s\| \leqslant 4D(s)$.

Proof. We consider first the inequality $D(s) \leqslant \|s\|$, which we will prove by induction on the number n of blocks of ones in the standard representation of s. By Lemma 2.3, it suffices to show that $D(s) \leqslant \sum_m^p |a_i| p_i$ for every representation of the form $s = \sum_m^p a_i 2^{-i}$, where $a_i \in \{0, \pm 1\}$ and $a_m \neq 0$. When $n = 1$, the desired inequality follows easily from the fact that $m \leqslant i_1$ (by Lemma 3.3). Suppose then that $D(s) \leqslant \|s\|$ whenever $1 \leqslant n < n_0$. Adopting the notation of the proof of Lemma 3.3, we know from that lemma that either (i) $\sum_m^\beta a_i 2^{-i} = B$ or (ii) $\sum_m^{\beta+1} a_i 2^{-i} = B + 2^{-(\beta+1)}$ and $a_{\beta+1} = -1$. Subtracting B from the standard representation of s and applying the inductive hypothesis to the remainder r, we find, respectively, that either

$$r = \sum_{\beta+1}^{p} a_i 2^{-i} \quad \text{and} \quad D(r) \leqslant \sum_{\beta+1}^{p} |a_i| p_i$$

or

$$r = 2^{-(\beta+1)} + \sum_{\beta+2}^{p} a_i 2^{-i} \quad \text{and} \quad D(r) \leqslant p_{\beta+1} + \sum_{\beta+2}^{p} |a_i| p_i = \sum_{\beta+1}^{p} |a_i| p_i.$$

Since $m \leqslant \alpha$, it follows in either case that

$$D(s) = p_\alpha + D(r) \leqslant p_m + \sum_{\beta+1}^{p} |a_i| p_i \leqslant \sum_{m}^{p} |a_i| p_i.$$

Thus $D(s) \leqslant \|s\|$, and the first inequality is proved.

To show that $\|s\| \leqslant 4D(s)$, we first rewrite the standard representation (2) for s as

$$s = \sum_{j=1}^{n} (2^{-(i_j-1)} - 2^{-k_j}).$$

Since $p_i \leqslant p_{i-1} \leqslant 2p_i$ for all i, it follows that

$$\|s\| \leqslant \sum_{j=1}^{n} (p_{i_j-1} + p_{k_j}) \leqslant \sum_{j=1}^{n} 2p_{i_j-1} \leqslant \sum_{j=1}^{n} 4p_{i_j} = 4D(s).$$

This concludes the proof of the lemma. ∎

We proceed now to the proof of Proposition 3.2. The proposition is trivial if $x \in S$, and thus we confine our attention to the case where $x \in \boldsymbol{R} \setminus S$, with standard representation given by (2). Suppose first that $D(x) < \infty$, and let s_m be the sum of the first m blocks of ones in the standard representation of x; that is, let

$$s_m = \sum_{j=1}^{m} \sum_{i=i_j}^{k_j} 2^{-i}.$$

When $m > k \geqslant 1$, we know from Lemma 3.4 that

$$\|s_m - s_k\| = \|\sum_{j=k+1}^{m} \sum_{i=i_j}^{k_j} 2^{-i}\| \leqslant 4 \sum_{j=k+1}^{m} p_{i_j},$$

and thus $\{s_m\}$ is a Cauchy sequence in $(S, \|\ \|)$. Therefore $\{s_m\}$ $\|\ \|$-converges to an element c of C, and the continuity of f assures that $f(c) = x$.

To prove the other half of the proposition, let $c \in f^{-1}(x)$ and choose a sequence $\{s_m\colon m \in N\} \subseteq S$ such that $\|s_m - c\| \to 0$. Then $|s_m - x| \to 0$ as well, and Lemma 3.1 allows us to replace $\{s_m\}$ with a subsequence and thus assume that $\{s_m\}$ is i_m-equivalent to x for all $m \in N$. Combining this observation with Lemma 3.4, we find that $\sum_{j=1}^{m} p_{i_j} \leqslant D(s_m) \leqslant \|s_m\|$ for all $m \in N$, whence we obtain $D(x) \leqslant \|c\| < \infty$ by letting $m \to +\infty$. ■

4. Proofs of theorems. In this section we prove Theorem 2.1 (iii) and Theorem 2.2. To prove the former, it suffices to show that every $\|\ \|$-Cauchy sequence $\{s_m\colon m \in N\} \subseteq S$ such that $|s_m| \to 0$ has a subsequence $\{s_{m(j)}\colon j \in N\}$ such that $\|s_{m(j)}\| \to 0$. Clearly it is enough to consider the case where $s_m > 0$ for all $m \in N$, and thus we can use Lemma 3.1 to obtain a subsequence $\{s_{m(j)}\}$, which we will re-name $\{t_j\colon j \in N\}$, such that $s_{m(j)}$ is j-equivalent to 0 for all $j \in N$. Now let $\varepsilon > 0$ be given, and choose $n \in N$ such that $p_j < \varepsilon/9$ and $\|t_j - t_n\| < \varepsilon/9$ for all $j \geqslant n$. If k is the largest value of i such that 2^{-i} has a non-zero coefficient in the standard representation for t_n, then the k-equivalence of t_k and 0 implies that $t_n - t_k$ is $(k-1)$-equivalent to t_n and that $D(t_n) \leqslant D(t_n - t_k) + p_k$. Using both inequalities in Lemma 3.4 and the fact that $k > n$, we obtain

$$\|t_n\| \leqslant 4D(t_n) \leqslant 4[D(t_n - t_k) + p_k] \leqslant 4[\|t_n - t_k\| + p_k] < 8\varepsilon/9,$$

and thus

$$\|t_j\| \leqslant \|t_j - t_n\| + \|t_n\| < \varepsilon \quad \text{for all } j \geqslant n.$$

Therefore $\|t_j\| \to 0$, and f is injective. ■

To prove Theorem 2.2, we will show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i). The implications (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) are trivial, and 2.1 (iii) makes it clear that (iii) $\Rightarrow$ (iv). It remains to prove that (i) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): Since $S \subseteq f(C)$, our task is to show that no element of $\boldsymbol{Q} \backslash S$ can be in $f(C)$, and for this it suffices (by 3.2) to show that $D(q) = \infty$ for every postive $q \in \boldsymbol{Q} \backslash S$. Now the standard representation for any such q will "end" with a repeating block of digits $a_b a_{b+1} \ldots a_c$ containing at least one 0 and at least one 1, and thus

$$q = \sum_{m}^{b-1} a_i 2^{-i} + \sum_{j=0}^{\infty} \sum_{i=0}^{k-1} a_{b+i}\, 2^{-(b+i+jk)},$$

where $a_i \in \{0, 1\}$ if $m \leqslant i < b$, $\{a_i\colon b \leqslant i \leqslant c\} = \{0, 1\}$, and $k = c-b+1$. Now every block of digits, except possibly the one corresponding to $j = 0$, contains a 1 that is preceded by a 0, and therefore $D(q) \geqslant \sum_{j=1}^{\infty} p_{c+jk}$. Since $p_{c+jk} \geqslant (1/k) \sum_{i=0}^{k-1} p_{c+jk+i}$ for all $j \in N$, it follows from the comparison test that $D(q) = \infty$ if $\sum_{1}^{\infty} p_i$ does not converge in $(\boldsymbol{R}, |\ |)$.

(v) $\Rightarrow$ (i): Using an argument that resembles the proof of 2.1 (iii), we will prove that $\|\ \|$ is not strictly stronger than $|\ |$ on S if $\sum_{1}^{\infty} p_i$ converges in $(\boldsymbol{R}, |\ |)$. To do this, it suffices to show that every sequence $\{s_i\colon i \in N\}$ in S such that $|s_i| \to 0$ has a subsequence $\{s_{i(j)}\colon j \in N\}$ such that $\|s_{i(j)}\| \to 0$. Since we may assume that $s_i \geqslant 0$ for all $i \in N$, Lemma 3.1 provides a subsequence $\{s_{i(j)}\colon j \in N\}$ such that $s_{i(j)}$ is j-equivalent to 0 and thus $\|s_{i(j)}\| \leqslant \sum_{j+1}^{\infty} p_i$ for all $j \in N$. That $\|s_{i(j)}\| \to 0$ then follows from the hypothesis that $\sum_{1}^{\infty} p_i < \infty$, and the proof of Theorem 2.2 is complete. ■

Before closing this section, we wish to note that both our definition of $\|\ \|$ and the strategy of using subseries of $\sum_{1}^{\infty} p_i$ to find $f(C)$ were suggested by an example in Section 3 of [1]. Indeed, that example would itself provide a negative answer to Mazur's question, were it not for the fact that the "norm" given there fails to satisfy the triangle inequality. Although an unpublished result of Erdös would apparently ([1], p. 207) allow the example in Section 2 of [1] to solve Problem 160, our construction has the advantage of being both simpler and self-contained.

5. Problem 160 and the definition of "connected". In [11] R. L. Wilder traced the development of the currently accepted definition of "connected", which is often (though perhaps not quite accurately) attributed to F. Hausdorff. As is well known, G. Cantor orginally proposed that a metric space (X, d) be called "connected" if, for every $x, x' \in X$ and for every $\varepsilon > 0$, there exists a finite set $\{x_1, \ldots, x_n\} \subseteq X$ such that $x = x_1$, $x_n = x'$, and $d(x_i, x_{i+1}) < \varepsilon$ if $1 \leqslant i < n$ ([11], p. 722). Although this criterion is equivalent to the usual one if (X, d) is compact, Example 31 in [9] (pp. 59–60) can be used to show that the two definitions are not equivalent for complete metric spaces.

When (X, d) is a topological group with left-invariant metric d, it is easy to prove that (X, d) is "Cantor-connected" if and only if it is generated by every neighbourhood of the identity. Thus for groups with left-invariant metrics, Mazur's Problem 160 asks whether a complete group that is Cantor-connected must also be Hausdorff-connected. As our Theorems 2.1 and 2.2 demonstrate, the answer is "no".

Remark. A. Gleason has pointed out to the author examples of Cantor-connected, Hausdorff-disconnected, complete metric spaces that are much simpler than the one we have drawn from [9]. Among these is the subset of $\boldsymbol{R}^2$ consisting of all (x, y) such that $x \neq 0$ and $y \geqslant 1/x^2$.

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DEPARTMENT OF COMPUTER SCIENCE, MATHEMATICS AND PHYSICS
ARKANSAS STATE UNIVERSITY
ARKANSAS 72467

and

SAINT EDMUND'S HOUSE
CAMBRIDGE CB3 0BN
ENGLAND

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