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MULTIPLE RADICAL THEORIES

BY

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If a non-trivial radical class $\mathscr{R}$ in the universal class of all associative rings is also a semi-simple class, the semi-simple class $\mathscr{S} = \{A \mid \mathscr{R}(A) = 0\}$ is not radical, while the radical class $\mathscr{C}$ for which $\mathscr{R} = \{A \mid \mathscr{C}(A) = 0\}$ is not semi-simple. In the present paper we answer the following question, inspired by a paper of Kurata [7]:

If $\mathscr{R}$ is a semi-simple radical class in some other universal class of rings, can $\mathscr{S}$ be radical and can $\mathscr{C}$ be semi-simple?

Before we proceed, we need to introduce some notation and terminology. If $\mathscr{R}$ is a radical class and $\mathscr{S}$ is the corresponding semi-simple class, we call $(\mathscr{R}, \mathscr{S})$ a *radical theory* (by analogy with the torsion theories for modules introduced by Dickson [2]). Extending this, and following the example of Kurata, who introduced n-fold torsion theories, we call an n-tuple $(\mathscr{R}_1, \ldots, \mathscr{R}_n)$ of classes of rings an *n-fold radical theory* if $(\mathscr{R}_j, \mathscr{R}_{j+1})$ is a radical theory for $j = 1, \ldots, n-1$. If there is an integer i such that $\mathscr{R}_{i+1} = \mathscr{R}_1$, the least such is called the *length* of $(\mathscr{R}_1, \ldots, \mathscr{R}_n)$; otherwise, $(\mathscr{R}_1, \ldots, \mathscr{R}_n)$ is said to have *length n*. Thus a semi-simple radical class is analogous to a TTF class as defined by Jans [6], while a 3-fold radical theory of length 2 is analogous (at this stage only formally, but — as we shall see later — in some important respects also) to a centrally splitting torsion theory in the sense of Bernhardt [1].

Kurata showed in [7] that an n-fold torsion theory ($n > 2$) must be either 3-fold of length 2, 3-fold of length 3 and not extendable to a 4-fold theory or 4-fold of length 4. There are several ways in which one might attempt to find analogues of this result for rings, depending on the kinds of universal classes in which one wanted to define radicals. We proceed as follows. All universal classes considered are varieties of (not necessarily associative) rings (no operators). A variety serving in this way is called *universal*. We show, in this setting, that the possibilities for an n-fold radical theory are exactly those for an n-fold torsion theory (Section 1). (It is to be expected that if other kinds of universal classes are used, the results may be different: Widiger and Wiegandt [13] have recently looked

at radical theory in the universal class of hereditarily artinian rings and have shown that there are homomorphically closed semi-simple classes which are not radical; this could not happen in a universal variety.)

In Section 2, we choose various universal varieties to demonstrate there the existence of n-fold radical theories of all "allowable" types. The parallels between 3-fold radical theories of length 2 and centrally splitting torsion theories are seen to be quite extensive. Such radical theories are also connected with the following question considered by Grätzer et al. [5]: If $\mathscr{U}$ and $\mathscr{V}$ are varieties of (universal) algebras, when is $\mathscr{U} \vee \mathscr{V} = \mathscr{U} \times \mathscr{V}$? Our characterization of 3-fold radical theories of length 2 also leads to a refinement of the structure theorem for autodistributive algebras obtained by Fiedorowicz [3].

There is a comparatively plentiful supply of 3-fold radical theories of length 3. In Section 3 we find some sufficient conditions on a universal variety for the absence of non-trivial 4-fold radical theories (and hence of 3-fold theories of length 2).

1. Possibilities. The results in this section, by and large, closely resemble some of Kurata's results in [7] for modules.

LEMMA 1.1. *Let $(\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3)$ be a 3-fold radical theory with $\mathscr{R}_1$ hereditary. Then $\mathscr{R}_1 \subseteq \mathscr{R}_3$.*

Proof. For any ring A in $\mathscr{R}_1$, we have $\mathscr{R}_2(A) \in \mathscr{R}_1 \cap \mathscr{R}_2 = \{0\}$, so $A \in \mathscr{R}_3$.

LEMMA 1.2. *Let $(\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3, \mathscr{R}_4)$ be a 4-fold radical theory with $\mathscr{R}_4$ homomorphically closed. Then $\mathscr{R}_4 = \mathscr{R}_2$.*

Proof. Since $\mathscr{R}_3$ is also homomorphically closed, for every ring A we have

$$A/\big(\mathscr{R}_2(A) + \mathscr{R}_3(A)\big) \in \mathscr{R}_3, \mathscr{R}_4,$$

so that $A = \mathscr{R}_2(A) + \mathscr{R}_3(A)$. In particular, if $A \in \mathscr{R}_4$, i.e. $\mathscr{R}_3(A) = 0$, then $A = \mathscr{R}_2(A)$, i.e. $A \in \mathscr{R}_2$, so $\mathscr{R}_4 \subseteq \mathscr{R}_2$. But $\mathscr{R}_2$ is hereditary, being a variety ([4], Theorem 1.5), so, by Lemma 1.1, $\mathscr{R}_2 \subseteq \mathscr{R}_4$.

COROLLARY 1.1. *There are no 4-fold radical theories of length 3.*

Proof. If $(\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3, \mathscr{R}_1)$ is a 4-fold radical theory, then $\mathscr{R}_1$ is homomorphically closed, so $\mathscr{R}_1 = \mathscr{R}_2$ which is impossible.

COROLLARY 1.2. *There are no 5-fold radical theories of length greater than 2.*

Proof. Let $(\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3, \mathscr{R}_4, \mathscr{R}_5)$ be a 5-fold radical theory with $\mathscr{R}_3 \neq \mathscr{R}_1$. Then $(\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3, \mathscr{R}_4)$ is a 4-fold theory with $\mathscr{R}_4$ homomorphically closed, so $\mathscr{R}_4 = \mathscr{R}_2$. But then $\mathscr{R}_3 = \mathscr{R}_1$.

COROLLARY 1.3. *For each $n > 4$, all n-fold radical theories have length 2.*

The preceding results combine to yield

THEOREM 1.1. *A multiple radical theory must be of one of the following kinds*:

(i) 3-*fold of length* 2,

(ii) 3-*fold of length* 3, *not extendable to* 4-*fold*,

(iii) 4-*fold of length* 4.

2. Examples and an application. In this section we show that all possibilities for multiple radical theories listed in Theorem 1.1 can be realized. The 3-fold radical theories of length 2 turn out to have connections with a question in universal algebra considered in [5] and we apply them to get a refinement of the structure theory for autodistributive algebras presented in [3].

THEOREM 2.1. *The following conditions are equivalent for subclasses* $\mathscr{U}$ *and* $\mathscr{V}$ *of a universal variety* $\mathscr{W}$:

(i) $(\mathscr{U}, \mathscr{V}, \mathscr{U})$ *is a 3-fold radical theory.*

(ii) $\mathscr{U}$ *and* $\mathscr{V}$ *are varieties and every ring* $R \in \mathscr{W}$ *has a unique representation* $R = A \oplus B$ *with* $A \in \mathscr{U}$ *and* $B \in \mathscr{V}$.

(iii) $\mathscr{U}$ *and* $\mathscr{V}$ *are varieties,* $\mathscr{U} \cap \mathscr{V} = \{0\}$ *and every ring* $R \in \mathscr{W}$ *has a representation* $R = A \oplus B$ *with* $A \in \mathscr{U}$ *and* $B \in \mathscr{V}$.

Proof. (i) $\Rightarrow$ (ii). Both $\mathscr{U}$ and $\mathscr{V}$ are semi-simple radical classes, and hence varieties ([4], Theorem 1.5). Furthermore, we have, for each $R \in \mathscr{W}$,

$$\mathscr{U}(R) = \bigcap \{K \lhd R \mid R/K \in \mathscr{V}\} \quad \text{and} \quad \mathscr{V}(R) = \bigcap \{K \lhd R \mid R/K \in \mathscr{U}\}.$$

Then

$$R/[\mathscr{U}(R) + \mathscr{V}(R)] \in \mathscr{V} \cap \mathscr{U} = \{0\},$$

so $R = \mathscr{U}(R) + \mathscr{V}(R)$, while $\mathscr{U}(R) \cap \mathscr{V}(R) \in \mathscr{U} \cap \mathscr{V}$, so $\mathscr{U}(R) \cap \mathscr{V}(R) = 0$, and thus $R = \mathscr{U}(R) \oplus \mathscr{V}(R)$ for each $R \in \mathscr{W}$. If now $R = A \oplus B$ with $A \in \mathscr{U}$ and $B \in \mathscr{V}$, then $A \subseteq \mathscr{U}(R)$, while since $R/A \cong B \in \mathscr{V}$, we have $\mathscr{U}(R) \subseteq A$, so that $A = \mathscr{U}(R)$. Similarly, $B = \mathscr{V}(R)$, and this establishes condition (ii).

(ii) $\Rightarrow$ (iii). If $R \in \mathscr{U} \cap \mathscr{V}$, then since $R \oplus 0 = R = 0 \oplus R$, we have $R = 0$.

(iii) $\Rightarrow$ (i). Assume that $R \in \mathscr{W}$ has an ideal I such that both I and R/I are in $\mathscr{U}$. Then $R = A \oplus B$, where $A \in \mathscr{U}$ and $B \in \mathscr{V}$. We have

$$I/I \cap A \cong (I + A)/A \subseteq R/A \cong B \in \mathscr{V},$$

while $I/I \cap A \in \mathscr{U}$ since $I \in \mathscr{U}$. Hence $I \cap A = I$, i.e. $I \subseteq A$. But then B, as a homomorphic image of R/I, is in $\mathscr{U}$, as well as $\mathscr{V}$, so $B = 0$, and thus $R \cong A \in \mathscr{U}$. Thus $\mathscr{U}$ is closed under extensions. By Theorem 1.4 of [4], $\mathscr{U}$ is a radical class. If T is a ring in $\mathscr{V}$, then $\mathscr{U}(T) \in \mathscr{U} \cap \mathscr{V} = \{0\}$, so $\mathscr{U}(T) = \{0\}$. On the other hand, any ring S is of the form $K \oplus L$ with $K \in \mathscr{U}$ and $L \in \mathscr{V}$, and then

$$\mathscr{U}(S) = \mathscr{U}(K) \oplus \mathscr{U}(L) = K \oplus \mathscr{U}(L) = K.$$

Thus, if $\mathscr{U}(S) = 0$, then $S\ (=L) \in \mathscr{V}$. This proves that $(\mathscr{U}, \mathscr{V})$ is a radical theory. Similarly, $(\mathscr{V}, \mathscr{U})$ is a radical theory.

COROLLARY 2.1. *Let $\mathscr{U}$ and $\mathscr{V}$ be as in Theorem 2.1, let F be a free $\mathscr{W}$-ring on $\aleph_0$ generators, and let I and J be the T-ideals defining $\mathscr{U}$ and $\mathscr{V}$, respectively. Then $F = J \oplus I$.*

Grätzer et al. (see [5], Theorem 1) have given some sufficient conditions on a pair of varieties $\mathscr{U}$, $\mathscr{V}$ for them to satisfy condition (ii) of Theorem 2.1, and Lee Sin-Min [8] has shown that these hold when $\mathscr{U}$ is the class of zero rings and $\mathscr{V}$ is generated by a finite set of finite associative fields. From Theorem 2 of [8] we get

THEOREM 2.2. *Let $\mathscr{W}$ be the variety of associative rings generated by a finite set of finite fields and all zero rings. Then $(\mathscr{U}, \mathscr{V}, \mathscr{U})$ is a 3-fold radical theory, where $\mathscr{U}$ is the class of zero rings and $\mathscr{V}$ the class of idempotent rings (in $\mathscr{W}$).*

Our second example concerns the *autodistributive rings* — the rings of characteristic 2 in which multiplication is distributive over itself, i.e. which satisfy the identities

$$2x = 0, \quad x(yz) = (xy)(xz), \quad (xy)z = (xz)(yz).$$

These rings were investigated by Fiedorowicz [3]. (It might be more appropriate to call a ring *autodistributive* if it satisfies the second and the third of the above-given identities. It follows from Theorem 1 of [3] that any 2-torsion-free ring satisfying these identities is nilpotent of index 3; the algebras over the field of two elements are studied in [3].)

THEOREM 2.3. *Let $\mathscr{W}$ be the variety of autodistributive rings, and let $\mathscr{N}$ and $\mathscr{E}$ be the varieties in $\mathscr{W}$ defined by the identities $x(yz) = 0 = (xy)z$ and $x^2 = x$, respectively. Then $(\mathscr{N}, \mathscr{E}, \mathscr{N})$ is a 3-fold radical theory.*

Proof. By Theorems 8, 9 and 10 of [3], $\mathscr{N}$ is a radical class with $\mathscr{E}$ as its semi-simple class. It follows that $\mathscr{E}$ is a radical class ([4], Theorem 1.5). We show that

$$\mathscr{E}(A) = \{a \in A \mid a^2 = a\} \quad \text{for all } A \in \mathscr{W}.$$

If $a = a^2$ and $b = b^2$, then $a = a^3$ and $b = b^3$, so, by Theorem 7 of [3],

$$(a+b)^3 = a^3+b^3 = a+b \quad \text{and} \quad (a+b)^4 = a^4+b^4 = a+b,$$

whence

$$(a+b)^2 = (a+b)(a+b) = (a+b)^3(a+b) = (a+b)^4 = a+b$$

(by Theorem 6 of [3], A is power-associative). If now a and c are in A with $a^2 = a$, then we have

$$(ac)^2 = (ac)(ac) = a^2c = ac, \quad (ca)^2 = (ca)(ca) = ca^2 = ca.$$

Thus the set of idempotent elements is an ideal of A, and so must coincide with $\mathscr{E}(A)$.

Hence the semi-simple class corresponding to $\mathscr{E}$ is

$$\mathscr{S} = \{A \in \mathscr{W} \mid a \in A, a^2 = a \Rightarrow a = 0\}.$$

If $s \in A \in \mathscr{S}$, then, by Theorem 5 of [3], $(s^3)^2 = s^6 = s^3$, so $s^3 = 0$ and A is a nil. Thus $\mathscr{S} \subseteq \mathscr{N}$. Clearly, $\mathscr{N} \subseteq \mathscr{S}$.

COROLLARY 2.2. *Every autodistributive ring is uniquely expressible as a direct sum of a nilpotent ring of index* 3 *and a ring in which all elements are idempotent.*

All of the above remains unchanged if we alter our point of view and consider algebras over the field of two elements. Thus Corollary 2.2 augments the structure theory in [3].

Let $\mathscr{R}$ be a semi-simple radical class of associative rings, and let $(\mathscr{C}, \mathscr{R})$ and $(\mathscr{R}, \mathscr{S})$ be the corresponding radical theories. Then $\mathscr{C}$ and $\mathscr{S}$ contain all zero rings, while all semi-simple radical classes consist of idempotent rings (see, e.g., [11]). It follows that $(\mathscr{C}, \mathscr{R}, \mathscr{S})$ is a non-extendable 3-fold radical theory of length 3.

THEOREM 2.4. *Let $\mathscr{W}$ be the universal variety of associative rings. Then in $\mathscr{W}$ there are* 3*-fold radical theories and none of them is extendable.*

We complete our survey by exhibiting a 4-fold radical theory of length 4.

THEOREM 2.5. *Let $\mathscr{W}$ be the universal variety consisting of all associative extensions of zero rings by (associative) boolean rings. In $\mathscr{W}$, let $\mathscr{I}$, $\mathscr{Z}$ and $\mathscr{E}$ be the classes of idempotent, zero and boolean rings, respectively. Then there exists a class $\mathscr{S}$ such that $(\mathscr{I}, \mathscr{Z}, \mathscr{E}, \mathscr{S})$ is a* 4*-fold radical theory of length* 4.

Proof. $\mathscr{E}$ is a semi-simple radical class in the universal variety of all associative rings, and hence in $\mathscr{W}$. Let $\mathscr{S}$ be the semi-simple class in $\mathscr{W}$ corresponding to $\mathscr{E}$ as a radical class. Now, every ring R in $\mathscr{W}$ has an ideal I such that $I^2 = 0$ and $R/I \in \mathscr{E}$. The sum of all ideals of R which belong to $\mathscr{Z}$ is a nil associative ring and belongs to $\mathscr{W}$, so it is in $\mathscr{Z}$. It follows that $\mathscr{Z}$ is a radical class. Clearly, $\mathscr{Z}(A) = 0$ for some ring A in $\mathscr{W}$ if and only if $A \in \mathscr{E}$. Arguing as in Theorem 1 of [9] or in Theorem 1 of [10], we can prove that $\mathscr{Z}$ is also a semi-simple class. The corresponding radical class is then clearly $\mathscr{I}$. Thus $(\mathscr{I}, \mathscr{Z}, \mathscr{E}, \mathscr{S})$ is a 4-fold radical theory. Let H be the field with two elements, and G a one-dimensional vector space over H. Let T be the ring with additive group $G \oplus H$ and multiplication given by

$$(g_1, h_1)(g_2, h_2) = (h_1 g_2 + h_2 g_1, h_1 h_2).$$

Then T is in $\mathscr{W}$, is idempotent, but is not boolean. Hence $\mathscr{I} \neq \mathscr{E}$, and the theorem is proved.

3. Impossibilities. In order to produce most of the examples in the previous section we had to look at universal varieties quite different from those in which radical theory is generally developed, and in the more familiar class of all associative rings we saw that 4-fold radical theories do not exist. In this section we generalize the latter result to a wide range of universal varieties.

THEOREM 3.1. *Let $\mathscr{W}$ be a universal variety containing all zero rings and such that*

$$\bigcap_n F^n = 0$$

for all free rings F in $\mathscr{W}$. Then $\mathscr{W}$ has no non-trivial 4-fold radical theories.

Proof. Suppose that $(\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3, \mathscr{R}_4)$ is a non-trivial 4-fold radical theory, i.e. none of $\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3, \mathscr{R}_4$ is $\mathscr{W}$. Then $\mathscr{R}_2$ and $\mathscr{R}_3$ are extension-closed varieties, so, by Corollary 1.9 of [4], they cannot contain any zero rings different from 0. Thus, if $A^2 = 0$, we have $\mathscr{R}_2(A) = 0$, i.e. $A \in \mathscr{R}_3$, but $\mathscr{R}_3(A) = 0$, so $A = 0$, contrary to the hypotheses on $\mathscr{W}$.

Note that the variety of associative rings satisfies the hypotheses of Theorem 3.1, but has 3-fold radical theories.

If $(\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3, \mathscr{R}_4)$ is a 4-fold radical theory, then $\mathscr{R}_2$ is a *strongly hereditary strict* radical class (see [12] for this terminology). Hence in universal varieties where classes of the latter kind are rare, 4-fold radical theories are unlikely to occur. We conclude by investigating this matter for varieties between the associative and the power-associative rings.

Let P be a non-empty set of primes. For a universal variety $\mathscr{W}$, we denote by $\mathscr{W}_P$ the class of rings in $\mathscr{W}$ whose additive groups are direct sums of p-groups with $p \in P$.

THEOREM 3.2. *Let $\mathscr{W}$ be a universal variety of power-associative rings which contains all associative rings. The only non-trivial strongly hereditary strict radical classes in $\mathscr{W}$ are the classes $\mathscr{W}_P$.*

Proof. Let $\langle a \rangle$ denote the subring of a ring generated by a single element a. If $\mathscr{R}$ is a non-trivial strongly hereditary strict radical class and A is in $\mathscr{W}$, then A is in $\mathscr{R}$ if and only if $\langle a \rangle$ is in $\mathscr{R}$ for each $a \in A$. Also, each such $\langle a \rangle$ is associative. Let $\mathscr{R}_a$ denote the class of associative rings in $\mathscr{R}$. Then $\mathscr{R}_a$ is a non-trivial strongly hereditary strict radical class in the universal variety of associative rings. By a result of Stewart ([12], Proposition 4.1), there is a set P of primes such that $\mathscr{R}_a$ is the class of all associative rings in $\mathscr{W}_P$. If $r \in R \in \mathscr{R}$, then $\langle r \rangle \in \mathscr{R}_a \subseteq \mathscr{W}_P$. Hence $\mathscr{R} \subseteq \mathscr{W}_P$. On the other hand, if $t \in T \in \mathscr{W}_P$, then $\langle t \rangle \in \mathscr{R}_a \subseteq \mathscr{R}$, whence $T \in \mathscr{R}$, so $\mathscr{R} = \mathscr{W}_P$.

COROLLARY 3.1. *Let $\mathscr{W}$ be as in Theorem 3.2. Then $\mathscr{W}$ has no non-trivial 4-fold radical theories.*

Proof. By Theorem 3.2 and the remarks preceding it, any 4-fold radical theory must have the form $(\mathscr{R}_1, \mathscr{W}_P, \mathscr{R}_3, \mathscr{R}_4)$ and $\mathscr{W}_P$ must be closed under formation of direct products. But $\mathscr{W}_P$ contains the zero ring on the cyclic group of order p^n for each $p \in P$ and for each n, and the product of these has elements of infinite order.

Added in proof. After the submission of this paper, there appeared a paper by T. Kepka (*On a class of non-associative rings*, Commentationes Mathematicae Universitatis Carolinae 18 (1977), p. 531-540) in which our Corollary 2.2 can also be found.

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