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# ON SIMULTANEOUS BLUMBERG SETS

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**1. Introduction.** Throughout the paper a space means a topological space and we do not assume the continuity of functions. For any $A \subset X$, the closure of $A$ and the interior of $A$ are denoted by $\mathrm{Cl}\,A$ and $\mathrm{Int}\,A$, respectively. Given a function $f: X \to Y$, denote its set of continuity by $C(f) = \{x \in X \mid f \text{ is continuous at } x\}$.

A function $f: X \to Y$ is called *quasi-continuous at a point* $x \in X$ ([7], p. 39) if for any open sets $A \subset X$ and $H \subset f(X)$, where $x \in A$ and $f(x) \in H$, we have $A \cap \mathrm{Int} f^{-1}(H) \neq \emptyset$. A function $f: X \to Y$ is called *quasi-continuous* if it is quasi-continuous at each point $x$ of $X$.

A function $f: X \to Y$ is called *somewhat continuous* if for each open set $V \subset Y$ the condition $f^{-1}(V) \neq \emptyset$ implies $\mathrm{Int} f^{-1}(V) \neq \emptyset$ (see [4], p. 6).

It can be easily verified that any quasi-continuous function is somewhat continuous.

A space $X$ is said to be a *Baire space* ([2], p. 75) if every non-empty open set in $X$ is of second category.

Let $f$ be a function from a space $X$. We say that $X$ is a *Blumberg space for $f$* ([11], Definition 3) if there exists a dense subset $D$ of $X$ such that the partial function $f|D$ is continuous. Such a set $D$ is called a *Blumberg set for $f$*.

A set $D$ is called a *full Blumberg set for $f$* ([11], Definition 4) if $D$ is a Blumberg set for $f$ and, for every open set $A \subset X$, the set $f(D \cap A)$ is dense in $f(A)$.

Let $f: X \to Y$ be a bijection. A set $D$ in $X$ is a *simultaneous Blumberg set for $f$* ([9], p. 452) if $D$ is a Blumberg set for $f$ and $f(D)$ is a Blumberg set for $f^{-1}$.

Given a family $F = \{f_i \mid f_i: X \to Y \text{ is a bijection}, i \in I\}$, a set $D$ in $X$ is called a *simultaneous Blumberg set for $F$* if $D$ is a simultaneous Blumberg set for each $f_i$, $i \in I$.

As the most important results of this paper we consider the Theorem and Corollary 3 in Section 3.

**2. Preliminary lemmas and propositions.**

LEMMA 1. *Let $X$ be a Baire space, let $Y$ be a second countable space, and let $f\colon X \to Y$ be quasi-continuous. Then $C(f)$ contains a dense $G_\delta$-subset of $X$.*

The lemma follows easily from Proposition 2 of [3], p. 985.

PROPOSITION 1. *Let $X$ and $Y$ be topological spaces. Let $f\colon X \to Y$ be a somewhat continuous bijection with the somewhat continuous inverse $f^{-1}\colon Y \to X$. If $G$ is a dense subset of $X$ such that $G \subset \mathrm{Cl}\,\mathrm{Int}G$, then* $\mathrm{Cl}\,\mathrm{Int}f(G) = Y$.

Proof. Let us assume that $\mathrm{Int}f(G)$ is not dense in $Y$. Therefore, there exists a non-empty open set $B$ of $Y$ such that $B \cap \mathrm{Int}f(G) = \emptyset$. Then it follows from somewhat continuity of $f$ that $A = \mathrm{Int}f^{-1}(B) \neq \emptyset$. Since

$$X = \mathrm{Cl}G \subset \mathrm{Cl}\,\mathrm{Cl}\,\mathrm{Int}G = \mathrm{Cl}\,\mathrm{Int}G,$$

$\mathrm{Int}G$ is dense in $X$. Thus $G' = A \cap \mathrm{Int}G \neq \emptyset$. Now, $f^{-1}$ is somewhat continuous, and so $\mathrm{Int}(f^{-1})^{-1}\, G' = \mathrm{Int}f(G') \neq \emptyset$. Clearly,

$$\mathrm{Int}f(G') = \mathrm{Int}f(A \cap \mathrm{Int}G) \subset \mathrm{Int}f(G).$$

On the other hand,

$$\mathrm{Int}f(G') \subset \mathrm{Int}f(A) = \mathrm{Int}f\big(\mathrm{Int}f^{-1}(B)\big) \subset \mathrm{Int}f\big(f^{-1}(B)\big) \subset B.$$

Thus we obtain $\emptyset \neq \mathrm{Int}f(G') \subset B \cap \mathrm{Int}f(G)$, a contradiction.

Note that the set of continuity $C(f)$ of a somewhat continuous function need not be, in general, a dense subset of $X$ (see [11], Remark 1, p. 34). Moreover, a somewhat continuous bijection need not be, in general, quasi-continuous (see [8], Proposition 1, p. 174). However, we have the following

COROLLARY 1. *Let $X$ be a Baire space, let $Y$ be a second countable space, and let $f\colon X \to Y$ be a quasi-continuous bijection with quasi-continuous $f^{-1}\colon Y \to X$. If $G$ is an open subset of $X$ such that $G$ contains a dense subset of $C(f)$, then* $\mathrm{Cl}\,\mathrm{Int}f(G) = Y$.

Proof. In fact, by Lemma 1, the set $C(f)$ is dense in $X$. Thus $G$ is dense in $X$. Since $G \subset \mathrm{Cl}G = \mathrm{Cl}\,\mathrm{Int}G$ and since every quasi-continuous function is somewhat continuous, the corollary follows easily from Proposition 1.

LEMMA 2. *If $Q_1, Q_2, \ldots$ are dense $G_\delta$-sets of a Baire space, then so is the set $Q_1 \cap Q_2 \cap \ldots$*

The proof is similar to that of Theorem 1 in [6], § 34, p. 417.

PROPOSITION 2. *Let $X$ and $Y$ be second countable Baire spaces and let $F$ be a countable family of quasi-continuous bijections from $X$ onto $Y$. If* *for each $f_n \in F$, $n \in N$, the inverse function $f_n^{-1}$ is quasi-continuous, then $F$ admits a simultaneous Blumberg set.*

Proof. By Lemmas 1 and 2, the set $\bigcap_{n=1}^{\infty} C(f_n^{-1})$ contains a dense $G_\delta$-set $D$ of $Y$. Let $\{G_i\}$ be a sequence of open subsets of $Y$ such that

$$D = \bigcap_{i=1}^{\infty} G_i.$$

Let $\mathrm{Int} f_n^{-1}(G_i) = E_{i,n}$. Then, in virtue of Corollary 1, for all $n \in N$ and for all $i \in N$ the set $E_{i,n}$ is dense in $X$. Thus

$$E = \bigcap_{i=1}^{\infty} \bigcap_{n=1}^{\infty} E_{i,n}$$

is a dense $G_\delta$-set of $X$ by Lemma 2.

Arguments similar to those at the beginning of the proof show that $\bigcap_{n=1}^{\infty} C(f_n)$ contains a dense $G_\delta$-set $D'$ of $X$. Put $H = E \cap D'$. Again, by Lemma 2, $H$ is a dense $G_\delta$-set of $X$.

To prove that $H$ is a simultaneous Blumberg set for $F$ we assume that $f_k$ is an arbitrary function from $F$. We have

$$H = E \cap D' \subset D' \subset \bigcap_{n=1}^{\infty} C(f_n) \subset C(f_k),$$

which shows that $H$ is a Blumberg set for $f_k$. Further, we obtain

$$\begin{aligned} f_k(H) &= f_k\Big(\bigcap_{i=1}^{\infty} \bigcap_{n=1}^{\infty} \mathrm{Int} f_n^{-1}(G_i) \cap D'\Big) \subset f_k\Big(\bigcap_{i=1}^{\infty} \bigcap_{n=1}^{\infty} \mathrm{Int} f_n^{-1}(G_i)\Big) \\ &\subset f_k\Big(\bigcap_{i=1}^{\infty} \bigcap_{n=1}^{\infty} f_n^{-1}(G_i)\Big) \subset f_k\Big(\bigcap_{i=1}^{\infty} f_k^{-1}(G_i)\Big) = f_k\Big(f_k^{-1}\Big(\bigcap_{i=1}^{\infty} G_i\Big)\Big) \\ &= \bigcap_{i=1}^{\infty} G_i = D \subset \bigcap_{n=1}^{\infty} C(f_n^{-1}) \subset C(f_k^{-1}). \end{aligned}$$

This shows that $f_k(H)$ is a Blumberg set for $f_k^{-1}$. Thus the proof is completed.

By our method we see that a simultaneous Blumberg set for $F$ is a $G_\delta$-subset of $X$. This generalizes some results of Neugebauer (see [9], p. 452).

The countability of $F$ is essential.

Example 1. Consider an uncountable family $F^*$ of quasi-continuous bijections $f_a$ of $[0, 1]$. Given $a \in (0, 1/2]$, define $f_a$ as follows:

for $a \in (0, 1/2)$,

$$f_a(x) = \begin{cases} x & \text{for } x \in [0, a] \cup [1-a, 1], \\ -x+1 & \text{for } x \in (a, 1-a); \end{cases}$$

for $a = 1/2$,

$$f_{1/2}(x) = \begin{cases} x & \text{for } x \in [0, 1/2), \\ -x+3/2 & \text{for } x \in [1/2, 1]. \end{cases}$$

There exists no simultaneous Blumberg set for $F^*$, since every point $x_0 \in (0, 1)$ is a point of discontinuity of a function of the family $F^*$, namely $f_{x_0}$ if $x_0 \leqslant 1/2$, or $f_{1-x_0}$ if $x_0 > 1/2$.

PROPOSITION 3. *Let $f$: $X \to Y$ be a quasi-continuous bijection. If $D$ is a simultaneous Blumberg set for $f$, then $f(D)$ is a full Blumberg set for $f^{-1}$.*

Proof. Put $D' = f(D)$. We will show that for each open subset $J$ of $Y$ the set $f^{-1}(D' \cap J)$ is dense in $f^{-1}(J)$. Take an open subset $K$ of $X$ such that $K \cap f^{-1}(J) \neq \emptyset$. If $x_0 \in K \cap f^{-1}(J)$, then $f(x_0) \in J$. Since $f$ is quasi-continuous at $x_0$, there exists a non-empty open set $U \subset K$ such that $f(U) \subset J$. The density of $D$ in $X$ implies $U \cap D \neq \emptyset$. But $U \subset K$ and $U \subset f^{-1}(J)$. Therefore

$$\emptyset \neq U \cap D = U \cap f^{-1}\big(f(D)\big) = U \cap f^{-1}\big(f(D) \cap J\big) \subset K \cap f^{-1}(D' \cap J).$$

Thus $K \cap f^{-1}(D' \cap J) \neq \emptyset$.

From Proposition 3 and Theorem 2 of [11] we obtain

COROLLARY 2. *Let $f$: $X \to Y$ be a quasi-continuous bijection, where $X$ is a regular space, and $Y$ is a Blumberg space for $f^{-1}$. If $D$ is a simultaneous Blumberg set for $f$, then $f^{-1}$ is quasi-continuous.*

Proof. In fact, $f$: $X \to Y$ is a quasi-continuous bijection and $D$ is a simultaneous Blumberg set for $f$. Thus, by Proposition 3, there exists a full Blumberg set for $f^{-1}$. Now, $f^{-1}$ is a function from a space $Y$, which is a Blumberg space for $f^{-1}$, into a regular space $X$. Hence Theorem 2 of [11], p. 34, can be applied, and thereby $f^{-1}$ is quasi-continuous.

The author is indebted very much to the reviewer for the following example showing that the regularity of $X$ is essential in Corollary 2.

Example 2. Take the reals with the natural topology as $Y$. As $X$ take the reals with the topology which is finer than the natural topology by assuming the set of irrationals to be open. The identity function from $X$ onto $Y$ admits a simultaneous Blumberg set (namely, the set of irrationals), but its inverse function is not quasi-continuous. The fact that $Y$ is a Blumberg space for $f^{-1}$ follows easily from Alas' statement quoted in Section 3.

## 3. Main theorem.

THEOREM. *Let $X$ and $Y$ be second countable Baire spaces, let $X$ be regular, let $F$ be a countable family of quasi-continuous bijections from $X$ onto $Y$, and let $Y$ be a Blumberg space for $f_n^{-1}$, for every $f_n \in F$. Then $F$ admits a simultaneous Blumberg set if and only if for every $f_n \in F$ the inverse function $f_n^{-1}$ is quasi-continuous.*

The Theorem follows easily from Proposition 2 and Corollary 2.

There exists an example ([9], Theorem 3, p. 454) of a function from $[0, 1]$ onto itself which is a quasi-continuous bijection and whose inverse is not quasi-continuous. Another one-to-one function which does not admit a simultaneous Blumberg set was given by Goffman [5].

Now we recall two definitions and a result due to Alas [1].

A *pseudobase* ([10], p. 157) for a space $X$ with the topology $T$ is a subset $P$ of $T$ such that every non-empty element of $T$ contains a non-empty element of $P$.

A subfamily $P$ of $T$ is called *$\sigma$-disjoint* ([12], p. 456) if

$$P = \bigcup \{P_n \colon n = 1, 2, \ldots\},$$

where each $P_n$ is a disjoint family.

STATEMENT (Alas). *Let $X$ be a Hausdorff, Baire space with a $\sigma$-disjoint pseudobase, let $Y$ be a Hausdorff second countable space, and let $f\colon X \to Y$ be a function. There exists a dense subset $D$ of $X$ such that the restriction of $f$ to $D$ is continuous.*

Every second countable space has a $\sigma$-disjoint pseudobase. Therefore, if $X$ is a Hausdorff, Baire, second countable space, $Y$ is a Hausdorff second countable space, and $f\colon X \to Y$ is a function, then $X$ is a Blumberg space for $f$. Thus we have a result which follows from the Theorem and Alas' statement:

COROLLARY 3. *Let $X$ and $Y$ be second countable, Hausdorff, Baire spaces, let $X$ be regular, and let $F$ be a countable family of quasi-continuous bijections from $X$ onto $Y$. Then $F$ admits a simultaneous Blumberg set if and only if for each $f_n \in F$, $n \in N$, the inverse function $f_n^{-1}$ is quasi-continuous.*

COROLLARY 4 ([9], Theorem 2, p. 452). *Let $f$ be a quasi-continuous bijection from the unit interval onto itself. Then $f$ admits a simultaneous Blumberg set if and only if $f^{-1}$ is quasi-continuous.*

PROBLEM (**P 1234**). Does the Theorem remain true if the requirements on $X$ or $Y$ to be Baire spaces are omitted? I conjecture that the answer is negative.

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