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AUTOMORPHISMS OF DIFFERENTIAL GROUPS

BY

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The aim of this note is to determine all automorphisms of the differential group L_n^r. The problem arises only if $r \geqslant 2$, because L_n^1 is the linear group $GL(n)$ and its automorphisms are known. We shall see that, unlike the linear group, the differential groups of higher order admit only regular automorphisms being simultaneously Lie group automorphisms.

Recall that *differential group* L_n^r is a group determined by all invertible r-jets belonging to $I(R^n, R^n)$ with a source and butt 0. The composition rule of jets is the group multiplication in L_n^r. We give this group the structure of a Lie group by associating with every jet $j^r(f)$ a sequence of parameters

$$A_{a_1}^{a} = \frac{\partial f^a}{\partial \xi^{a_1}}, \ldots, A_{a_1 \ldots a_r}^{a} = \frac{\partial^r f^a}{\partial \xi^{a_1} \ldots \partial \xi^{a_r}}, \quad \det(A_{a_1}^{a}) \neq 0,$$

partial derivatives being taken at 0. In this way L_n^r becomes a Lie group of dimension

$$N = n\left[\binom{n+r}{r} - 1\right],$$

whose elements $L = (A_{a_1}^{a}, \ldots, A_{a_1 \ldots a_r}^{a})$ are subject to the multiplication law

$$(1) \qquad \begin{cases} C_{a_1}^{a} = A_{\beta}^{a} B_{a_1}^{\beta}, \\ C_{a_1 a_2}^{a} = A_{\beta_1 \beta_2}^{a} B_{a_1}^{\beta_1} B_{a_2}^{\beta_2} + A_{\beta}^{a} B_{a_1 a_2}^{\beta}, \\ C_{a_1 a_2 a_3}^{a} = A_{\beta_1 \beta_2 \beta_3}^{a} B_{a_1}^{\beta_1} B_{a_2}^{\beta_2} B_{a_3}^{\beta_3} + 3 A_{\beta_1 \beta_2}^{a} B_{(a_1}^{\beta_1} B_{a_2 a_3)}^{\beta_2} + A_{\beta}^{a} B_{a_1 a_2 a_3}^{\beta}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ C_{a_1 \ldots a_s}^{a_1} = A_{\beta_1 \ldots \beta_s}^{a} B_{a_1}^{\beta_1} \ldots B_{\beta_s}^{\beta_s} + A_{\beta}^{a} B_{a_1 \ldots a_s}^{\beta} + \\ \qquad + \sum_{p=2}^{s-1} A_{\beta_1 \ldots \beta_p}^{a} W^{\beta_1 \ldots \beta_p}(B_{a_1}^{a}, \ldots, B_{a_1 \ldots a_p}^{a}), \end{cases}$$

where $W^{\beta_1 \ldots \beta_p}$ are polynomials.

Fix $r \geqslant 2$ and $n \geqslant 1$, and write for $p = 2, \ldots, r$

$$H_p = \{(\delta^a_{\beta_1} 0, \ldots, 0, A^a_{a_1 \ldots a_p}, \ldots, A^a_{a_1 \ldots a_r})\},$$
$$G_p = \{(A^a_\beta, A^a_{a_1 a_2}, \ldots, A^a_{a_1 \ldots a_{p-1}}, 0, \ldots, 0)\}.$$

The following facts are stated in [2]:

(i) Every H_p is an invariant subgroup of L^r_n.

(ii) If a group G is a subgroup of L^r_n and contains a set G_p, then $G = L^r_n$.

Moreover, it is easy to show that

(iii) For every $p > 1$, the elements of L^r_n admit the factorization

$$L = (A^a_\beta, \ldots, A^a_{a_1 \ldots a_{p-1}}, 0, \ldots, 0)(\delta^a_\beta, 0, \ldots, 0, A^{-1a}_{\beta} A^\beta_{a_1 \ldots a_p} \ldots). \tag{3}$$

(iv) If $r > 2$, then H_r is the centre of H_2.

Notice also that subgroup H_r is additive and its essential parameters $A^a_{a_1 \ldots a_r}$ are subject to the following composition law:

$$C^a_{a_1 \ldots a_r} = A^a_{a_1 \ldots a_r} + B^a_{a_1 \ldots a_r}. \tag{4}$$

Finally, note that if $L_0 = (A^a_\beta, 0, \ldots)$ and $L = (B^a_\beta, B^a_{a_1 a_2}, \ldots)$, then we get for $C = L_0^{-1} L L_0$

$$C^a_{a_1 \ldots a_p} = A^{-1a}_{\beta} B^\beta_{\beta_1 \ldots \beta_p} A^{\beta_1}_{a_1} \ldots A^{\beta_p}_{a_p} \quad \text{for all } p = 2, \ldots, r. \tag{5}$$

We use short notation

$$A_s = (A^a_{a_1 \ldots a_s}), \quad A = A_1, \quad L = (A, A_2, \ldots, A_r);$$

$$h(L) = \big(h_1(L), \ldots, h_r(L)\big) \quad \text{for any automorphism } h.$$

Formula (5) can be abbreviated to

$$C_p = A^{-1} B_p A \ldots A \quad (A \text{ repeated } p \text{ times}). \tag{5'}$$

Parameters $A^a_{a_1 \ldots a_s}$ will be said to be *of order* s.

Recall also that the *characteristic subgroup* of a group G is defined as a subgroup being mapped onto itself by every automorphism of G.

The following theorem is the main result of this paper:

THEOREM. *For $r = 2$ any automorphism of the differential group L^r_n is the composite of an inner automorphism and of the automorphism*

$$h: \begin{cases} A^a_\beta \to A^a_\beta, \\ A^a_{\beta\gamma} \to A^a_{\beta\gamma} - \dfrac{2}{n+1} A^a_{(\beta} A^\sigma_{\gamma)\varrho} A^{-1\varrho}_{\sigma} + c \dfrac{2}{n+1} A^a_{(\beta} A^\sigma_{\gamma)\varrho} A^{-1\varrho}_{\sigma}, \end{cases} \tag{6}$$

where c is a non-zero constant. If $r > 2$, the only automorphisms of the group are the inner ones.

LEMMA 1. *If a subgroup H_p is characteristic in L_n^r and $h = (h_1, \ldots, h_r)$ is an automorphism, then its part $h' = (h_1, \ldots, h_{p-1})$ does not depend on parameters of order $\geqslant p$ and, as a function of variables $A_1, \ldots, A_{p-1}$, it defines an automorphism of the group L_n^{p-1}.*

Proof. By the assumption and statement (iii), h' depends only on parameters up to order $p-1$. On the other hand, we have $h' = \pi \circ h$, where π is the natural projection

$$L_n^r \to L_n^{p-1}; \ (A_1, \ldots, A_r) \to (A_1, \ldots, A_{p-1}), \qquad p \leqslant r.$$

Also, h' is a homomorphism, because both factors are. H_p being characteristic, automorphisms h and its inverse h^{-1} leave it invariant. This implies that h' restricted to G_p is a bijection onto L_n^{p-1}.

LEMMA 2. *Let*

$$h(A, 0, \ldots) = \big(G(A), H_2(A), \ldots, H_r(A)\big), \qquad A \in GL(n). \tag{7}$$

Then the matrix function G is one of the forms

$$G(A) = \varphi(\det A) C^{-1} A C, \tag{8}$$

or

$$G(A) = \varphi(\det A) C^{-1} (A^T)^{-1} C, \tag{9}$$

where φ is a scalar multiplicative function, and C is a constant non-singular matrix.

Proof. Since $(A, 0, \ldots)(B, 0, \ldots) = (AB, 0, \ldots)$, the function G satisfies the equation $G(AB) = G(A)G(B)$, and so G is an endomorphism of $GL(n)$. Consequently, it must be of form (8) or (9), or $G(A) = \varphi(\det A)$ (cf. [1]). We have to eliminate the latter case.

In the latter case $G(A) = E$ for every unimodular matrix $A \in SL(n)$. Let H_k be the first function of (7) non-vanishing on $SL(n)$. It satisfies the functional equation

$$H_k(AB) = G(A) H_k(B) + H_k(A) G(B) \ldots G(B), \tag{10}$$

the brief notation (5′) being used. In the considered case (10) takes the form

$$H_k(AB) = H_k(A) + H_k(B). \tag{11}$$

This means that H_k is commutative on $SL(n)$, so it must vanish on its commutator supgroup. But it is the group $SL(n)$ itself. We conclude that if G is constant on $SL(n)$, then all functions $H_2, \ldots, H_r$ of (7) must vanish. Hence h is constant on $SL(n)$ which is impossible. This completes the proof.

LEMMA 3. *Subgroup H_2 is characteristic.*

Proof. Let h be an automorphism. For any element $(E, 0, \ldots, 0, X)$, $X = (X^a_{a_1 \ldots a_r})$, belonging to H_r put

$$h(E, 0, \ldots, 0, X) = \big(K(X), \ldots\big). \tag{12}$$

From (4) and (5) it follows easily that K must fulfil equations

$$K(X+Y) = K(X)K(Y) \quad \text{(matrix product)}, \tag{13}$$

$$K(A^{-1}XA \ldots A) = G^{-1}(A)K(X)G(A) \tag{14}$$

for any X and every matrix $A \in GL(n), G(A)$ being defined by (7).

Equation (13) says that the matrix family $\{K(X)\}$ is commutative. Since for two commutative matrices the eigenvalues of their product are the products of their eigenvalues, we have, for any $k(X)$ belonging to the spectrum of $K(X)$ (¹),

$$k(X+Y) = k(X)k(Y) \tag{15}$$

and, by (14),

$$k(A^{-1}XA \ldots A) = k(X). \tag{16}$$

Setting $A = aE$, $a^{r-1} = 2$, into (16), we get $k(2X) = k(X)$. But by (15) there is $k(2X) = k^2(X)$, hence $k(X) \equiv 1$, because K is non-singular. Hence we proved that all eigenvalues of any matrix $K(X)$ are unities.

The commutative family $\{K(X)\}$ has a non-null invariant subspace V spanned by all its common eigenvectors. For every $v \in V$ and X we have $K(X)v = v$. From (14),

$$G(A)K(A^{-1}XA \ldots A) = K(X)G(A). \tag{17}$$

By multiplying (17) with a v from V we get $G(A)v \in V$. It means that V is also an invariant subspace of the family $\{G(A)\}$, $A \in GL(n)$. But this family with $G(A)$ being of form (8) or (9) is irreducible and, consequently, V must be the full space of dimension n. But $K(X)v = v$ for any vector v means that K is the unit matrix.

Accordingly, taking into account (12), we can state that the parameters of order r do not influence the matrix parameters of the image, i.e., $h_1(A, A_2, \ldots, A_r) = \tilde{h}_1(A, A_2, \ldots, A_{r-1})$. Similarly, we can show that $\tilde{h}_1$ does not depend on A_{r-1}, and so on. Finally, we come to the conclusion that h_1 depends only on A. Therefore, $h(E, A_2, \ldots) = (E, \ldots)$, and this was to be shown.

LEMMA 4. *Subgroups $H_p, p = 2, \ldots, r$, are all characteristic.*

(¹) $k(X)$ can be established as an element $k_{ii}(X)$, if a basis is chosen in a way such that all matrices $K(X)$ are triangular.

Proof. Immediate calculation shows that subgroup H_r is the centre of H_2. Therefore, it is a characteristic subgroup of H_2, which, by lemma 3, is characteristic in L_n^r. Hence H_r is characteristic in L_n^r. Combining this fact with lemma 1, we infer easily that also H_{r-1} is a characteristic subgroup of L_n^r, and so on.

LEMMA 5. *To any automorphism h there is an inner automorphism g such that the composite $g\circ h$ is the identity on the set G_2, i.e.,*

$$g\circ h(A, 0, \ldots) = (A, 0, \ldots). \tag{18}$$

Proof. We start with (7). Let H_k be the first function which does not vanish on $GL(n)$. It satisfies equation (10) for any $A, B \in GL(n)$.

Set $B = tE$, $t > 0$. Then, according to (8) or (9), $G(B) = a(t)E$. We have $a(t) = a^2(\sqrt{t}) > 0$. The case $a(t) \equiv 1$ is impossible, because then it would be $h(tE, 0, \ldots) = \big(E, H_2(tE), \ldots\big)$ with $H_i(tE)$ non-vanishing simultaneously for $t \neq 1$, and with the inverse automorphism h^{-1} send a non--identity element from H_2 outside it, which contradicts lemma 3.

For the matrix B as defined above we have $AB = BA$ for any A. Making use of commutativity, we get from (10)

$$H_k(A)\big(a^k(t)-a(t)\big) = H_k(tE)\underbrace{G(A)\ldots G(A)}_{k \text{ times}}-G(A)H_k(tE). \tag{19}$$

Choose t_0 such that $b = a^k(t_0)-a(t_0) \neq 0$. Then, by (19),

$$H_k(A) = UG(A)\ldots G(A)-G(A)U, \tag{20}$$

where $U = 1/bH_k(tE)$. Denote by g_k the inner automorphism generated by the element $L = (E, 0 \ldots 0, U, 0 \ldots 0)$, U standing in place k. We have

$$\begin{aligned}(g_k\circ h)(A, 0, \ldots) &= L^{-1}\big(G(A), 0 \ldots 0, H_k(A), \ldots\big)L\\ &= \big(G(A), 0 \ldots 0, \tilde{H}_{k+1}(A), \ldots\big),\end{aligned}$$

i.e., the first non-vanishing function H, if exists, has now an index $\tilde{k} > k$. In this way, by imposing succesively new inner automorphism $g_{\tilde{k}}$, etc., if needed, we obtain an automorphism which maps $(A, 0, \ldots)$ into $\big(G(A), 0, \ldots\big)$.

Note also that, up to the inner automorphism generated by element $(C^{-1}, 0, \ldots)$, we can take $C = E$ in formulas (8) and (9).

Now we prove that $G(A) \equiv A$, provided $C = E$. By lemma 1, the couple $h' = (h_1, h_2)$ defines an automorphism of L_n^2. According to what is already proved, we may assume $h'(A, 0) = \big(G(A), 0\big)$. By lemma 3, $h'(E, X) = \big(E, F(X)\big)$, where $X = (X^a_{\beta\gamma})$ and $F = (F^a_{\beta\gamma})$. By imposing h' upon the equality

$$(A^{-1}, 0)(E, X)(A, 0) = (E, A^{-1}XAA) \quad \text{and} \quad (E, X)(E, Y) = (E, X+Y)$$

we obtain

$$F(A^{-1}XAA) = G^{-1}(A)F(X)G(A)G(A), \tag{21}$$

$$F(X+Y) = F(X)+F(Y). \tag{22}$$

Set $A = tE$. Then $G(A) = a(t)E$, where $a(t) = \varphi(t^n)t^\varepsilon$ ($\varepsilon^2 = 1$). From (21) we infer that $F(tX) = a(t)F(X)$.

Substituting it to (22), we easily get $a(t+s) = a(t)+a(s)$. Thus the function $a(t)$ is additive and multiplicative. Consequently, $a(t) \equiv t$. This implies that $\varphi(s) = s^{(1-\varepsilon)/n}$, and (21) takes the form

$$F(A^{-1}XAA) = A^{-1}F(X)AA \tag{23}$$

in the case of (8), and

$$F(A^{-1}XAA) = (\det A)^{2/n}B^{-1}F(X)BB, \qquad B = (A^T)^{-1}, \tag{24}$$

in the case of (9).

In the former case $G(A) \doteq A$, and in the latter case $G(A) = (\det A)^{2/n}(A^T)^{-1}$. In order to prove the lemma, we have to eliminate the latter case. Note that if $n = 2$, then both cases are equivalent, for there is a constant matrix C such that

$$(\det A)(A^T)^{-1} \equiv C^{-1}AC; \qquad \text{for instance } C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{25}$$

In terms of tensors, (24) means that F is a mapping of a symmetric tensor space $T^{(1,2)}$ of valence (1,2) into a space $T^{(2,1)}$ of symmetric tensor densities of valence (2,1) and weight $2/n$. Moreover, F is linear. It is easy to show that such a linear mapping, being compatible with the action of the linear group on the spaces in question, can exist non-trivially only if $n = 2$. This completes the proof of the lemma.

LEMMA 6. *Any automorphism of L_n^2, up to an inner automorphism, is of the form* (6).

Proof. Put $(A, X) = (A^\alpha_\beta, X^\alpha_{\beta\gamma})$, and let $h(E, X) = (E, F(X))$. As it was shown in the proof of the preceding lemma, function F must satisfy equation (23) and be linear. Thus F is a linear mapping of the space $T^{(1,2)}$ into itself, compatible with the tensor representation

$$X \to A^{-1}XAA \qquad (= \overset{-1}{A}{}^\alpha_\varrho X^\varrho_{\tau\sigma} A^\tau_\beta A^\sigma_\gamma). \tag{26}$$

But the tensor representation theory says (e.g., Weyl [3]) that the representation (26) is the direct sum of two irreducible representations whose invariant subspaces are

$$V_1 : X^\varrho_{\alpha\varrho} = 0, \qquad V_2 : X^\alpha_{\beta\gamma} - \frac{2}{n+1}\,\delta^\alpha_{(\beta} X^\varrho_{\gamma)\varrho} = 0. \tag{27}$$

Since F is a linear operator in $T^{(1,2)}$ commuting with the representation (26), it must break up into the direct sum of two linear operators acting in subspaces V_1 and V_2, respectively. Each of them must be a multiple of identity. This means that for every $X \in T^{(1,2)}$ with decomposition $X = Y+Z$ there is $F(X) = aY+bZ$, $Y \in V_1$ and $Z \in V_2$.

In terms of components

$$X^{\alpha}_{\beta\gamma} = a\left(X^{\alpha}_{\beta\gamma} - \frac{2}{n+1}\,\delta^{\alpha}_{(\beta}X^{\varrho}_{\gamma)\varrho}\right) + b\left(X^{\alpha}_{\beta\gamma} + \frac{2}{n+1}\,\delta_{(\beta}X^{\varrho}_{\gamma)\varrho}\right) \tag{28}$$

F being injective, constants a and b are different from 0. By an inner automorphism we may obtain $F(X) = Y+cZ$, $c \neq 0$. To get (25) notice that $(A, X) = (A, 0)(E, A^{-1}X)$, whence $h(A, X) = (A, 0)\big(E, F(A^{-1}X)\big) = \big(A, AF(A^{-1}X)\big)$.

LEMMA 7. *If $r > 2$, then the automorphism* (6) *can be extended to that of L^r_n if and only if it is the identity* (*i.e.*, $c = 1$).

Proof. Let $r = 3$. According to (1), we have $(E, X, 0)(E, Y, 0) = (E, X+Y, XY)$, where by XY we mean the set of arguments

$$(XY)^{\alpha}_{\alpha_1\alpha_2\alpha_3} = 3X^{\alpha}_{\beta(\alpha_1}Y^{\beta}_{\alpha_2\alpha_3)}. \tag{29}$$

Let $L \in H_2$. Then $h(L) = \big(E, h_2(L), h_3(L)\big)$. By (1), h_3 fulfils the equation

$$h_3(L_1L_2) = h_3(L_1)+h_3(L_2)+h_2(L_1)h_2(L_2) \tag{30}$$

(multiplication as above).

Let $L_1 = (E, X, 0)$ and $L_2 = (E, Y, 0)$. If $XY = YX$, then $L_1L_2 = L_2L_1$, and from (30) we get

$$h_2(L_1)h_2(L_2) = h_2(L_2)h_2(L_1). \tag{31}$$

In notation of the preceding proof we have $h_2(E, X, 0) = F(X) = Y+cZ$; Y, Z defined by (28).

Choose $X^{\alpha}_{\beta\gamma} = \Theta^{\alpha}v_{\beta}v_{\gamma}$ and $Y^{\alpha}_{\beta\gamma} = \tau^{\alpha}u_{\beta}u_{\gamma}$ such that $\Theta^{\varrho}u_{\varrho} = \tau^{\varrho}v_{\varrho} = 0$ and $\tau^{\varrho}u_{\varrho} = \Theta^{\varrho}v_{\varrho} = 1$.

A straightforward calculation shows that in this case $XY = YX$, but (31) can hold if and only if $c = 1$. It completes the proof for $r = 3$. If $r > 3$, then, by lemma 1, the triple (h_1, h_2, h_3) is an automorphism of L^3_n, and so, up to an inner automorphism, the couple (h_1, h_2) must be the identity.

LEMMA 8. *Let $r > 2$. Up to an inner automorphism, any automorphism $h = (h_1, \ldots, h_r)$ is uniquely determined by its part (h_1, h_2).*

Proof. It is sufficient to show h_r is uniquely determined if $h' = (h_1, \ldots, h_{r-1})$ is given. If we have it, we can apply lemma 1 and repeat it to h', and so on.

According to the multiplication law (1), h_r must be a solution of the functional equation

$$h_r(LL') = h_r(L)B \dots B + Ah_r(L') + W\big(h'(L), h'(L')\big), \tag{32}$$

where $L = (A, \dots)$ and $L' = (B, \dots)$. Let g be the difference of any two solutions of equation (32). By subtracting we obtain from (32)

$$g(LL') = g(L)B \dots B + Ag(L'). \tag{33}$$

For L, L' from H_2 and $A = B = E$ we have $g(LL') = g(L) + g(L')$.

Since g is an abelian function on H_2, it must vanish on its commutator supgroup. But this is H_3, because H_3 is the largest subgroup of H_2 such that the factor group H_2/H_3 is abelian, and which does not contain elements with $A_2 \neq 0$. Consequently, g vanishes on H_3.

Up to an inner automorphism, we can assume that g vanishes also on the set G_2, i.e., $g(A, 0, \dots) = 0$. What remains to show is that $g(X) = g(E, X, 0, \dots) = 0$. From (33) we obtain easily

$$g(X+Y) = g(X) + g(Y), \tag{34}$$

$$g(A^{-1}XAA) = A^{-1}g(X)A \dots A \quad (A \text{ repeated } r \text{ times}). \tag{35}$$

Set $A = 2E$ into (35). Then $g(2X) = 2^{r-1}g(X)$, where $r-1 > 1$. On the other hand, from (34), $g(2X) = 2g(X)$. It gives $g(X) = 0$. Taking all these altogether, we have $g(L) = 0$ on the whole group L_n^r.

Proof of the theorem. If $r = 2$, the theorem follows from lemma 6. Let $r = 3$. By lemmas 1 and 4 the couple (h_1, h_2) is an automorphism of L_n^2. Up to an inner automorphism of L_n^r, it has form (6), provided (18) holds. By lemma 7 it is identity, which, by lemma 8, can be extended to an automorphism of the whole group in a unique way, i.e., to the identity automorphism.

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