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A TECHNIQUE FOR RECONSTRUCTING DISCONNECTED GRAPHS

BY

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Aside from the Four Color Problem, the best known unsolved problem in graph theory is the so-called Reconstruction Problem. This problem originated in 1957 with a conjecture of Kelly [5] which states:

If G and H are two graphs of order $p \geqslant 3$ whose vertex sets are $V(G) = \{v_1, v_2, \ldots, v_p\}$ and $V(H) = \{u_1, u_2, \ldots, u_p\}$, and $G - v_i$ is isomorphic to $H - u_i$ (written $G - v_i = H - u_i$) for $i = 1, 2, \ldots, p$, then $G = H$. In 1960, Ulam [7], p. 29, stated this conjecture in a more general setting, using metric spaces; thus this conjecture is often referred to as the Kelly-Ulam Conjecture. In 1964, Harary [4] reformulated the conjecture. A graph G with $V(G) = \{v_1, v_2, \ldots, v_p\}$, $p \geqslant 3$, is said to be *reconstructable* if G is determined (uniquely) from the p subgraphs $G - v_i$, $i = 1, 2, \ldots, p$. Harary's version of the Kelly-Ulam Conjecture can now be given.

The reconstruction conjecture. Every graph of order at least three is reconstructable.

The fact that a graph G of order $p \geqslant 3$ is reconstructable does not necessarily imply the existence of a special technique to construct or display G from the p subgraphs $G - v$, $v \in V(G)$. In a finite number of steps, one can determine all graphs of order p (disregarding the question of isomorphism). For example, it would not be difficult to determine all possible p-by-p matrices which are the adjacency matrices of graphs (although it might be quite difficult to establish which matrices correspond to non-isomorphic graphs). Again, after a finite number of steps, the graph G could be located from among the graphs of order p. Certainly, this procedure is likely to be highly inefficient; thus even when a graph G or class of graphs G has been proved reconstructable, the problem always

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remains to determine the most efficient algorithm or procedure to display G from its subgraphs $G-v$, $v \in V(G)$. Another problem in this context is the following:

Given p graphs G_i, $i = 1, 2, \ldots, p$, determine whether there exists a graph G of order p whose vertices may be labelled $v_1, v_2, \ldots, v_p$ such that $G_i = G-v_i$ for $i = 1, 2, \ldots, p$. A discussion of this problem is given in O'Neil [6]. In this paper, we assume that a graph G always exists.

The class of graphs which we consider in this article is the class of disconnected graphs. Disconnected graphs are known to be reconstructable; an existence proof of this fact is given in [2]. In [1] and [4] procedures are presented which enable one to reconstruct a disconnected graph G from the subgraphs $G-v$, $v \in V(G)$, in many instances. However, in both cases the procedures are incomplete and without proof. It is the purpose here to present a convenient and relatively uncomplicated technique for constructing all disconnected graphs G from the subgraphs $G-v$, $v \in V(G)$.

In order to present this result, we begin with three lemmas due to Harary (cf. [3] and [4]).

LEMMA 1. *Let G be a graph with q edges and suppose $V(G) = \{v_1, v_2, \ldots, v_p\}$, where $p \geqslant 3$. Then it is possible to determine q and the degrees of the vertices v_i from the subgraphs $G-v_i$, $i = 1, 2, \ldots, p$.*

LEMMA 2. *Every connected graph of order at least two has two or more vertices which are not cut-vertices.*

LEMMA 3. *If G is a graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$, $p \geqslant 3$, then G is connected if and only if at least two of the subgraphs $G-v_i$ are connected.*

Reconstruction algorithm. We now proceed to the main result which is the description of a reconstruction algorithm for disconnected graphs of order $p \geqslant 3$.

Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$, $p \geqslant 3$. By Lemma 3, it is possible to determine whether G is connected from the subgraphs $G-v_i$, $i = 1, 2, \ldots, p$. By Lemma 1, it is possible to determine the degrees of the vertices v_i from the subgraphs $G-v_i$. If a vertex v_j has degree 0, i. e., if v_j is isolated, then G consists of $G-v_j$ together with one additional isolated vertex.

We henceforth assume that G has no isolated vertices. Let k ($\geqslant 2$) denote the number of components of G. Each subgraph $G-v_i$ has k components or more than k components depending on whether v_i is not a cut-vertex or is a cut-vertex of G. By Lemma 2, every component of G contains at least two vertices which are not cut-vertices of G. We now consider only those subgraphs $G-v_i$ having k components.

Among all subgraphs $G-v$ with k components, select one having a component of minimum order m. Suppose $G-u$ is a subgraph containing the component F of order m. We note that F is the only component of

$G-u$ having order m and that F is obtained by the removal of u from a component of G, i. e., $F = F_1-u$ for some component F_1 of G. Denote the remaining components of $G-u$ by $F_2, F_3, \ldots, F_k$. It thus follows that $k-1$ of the k components of G are $F_2, F_3, \ldots, F_k$, which are immediately discernible from $G-u$. Hence, to reconstruct G, it remains only to identify F_1. We now consider three cases, depending on the orders of the components F_i, $2 \leqslant i \leqslant k$.

Case 1. *Some component F_i, $2 \leqslant i \leqslant k$, has order at least $m+3$.* Assume there are r components of order $m+1$ among the components $F_2, F_3, \ldots$ $\ldots, F_k$. (It may occur that $r = 0$.) Select a subgraph $G-v_j$ with k components having $r+1$ components of order $m+1$. Thus v_j is necessarily a vertex of a component of G having order at least $m+3$. Therefore, all components of $G-v_j$ of order $m+1$ are components of G (one of which is F_1). Hence G consists of all components of order $m+1$ in $G-v_j$ together with all components among $F_2, F_3, \ldots, F_k$ having order greater than $m+1$.

Case 2. *All components F_i, $2 \leqslant i \leqslant k$, have order $m+2$.* We consider all subgraphs $G-v_i$ with k components having two components of order $m+1$. Necessarily, in each such subgraph $G-v_i$, one of the two components of order $m+1$ is F_1. If there is only one graph which occurs among the pairs of components having order $m+1$, then this graph is F_1, which completes the determination of G. Otherwise, assume that every pair consists of the same two (non-isomorphic) components, say F' and F''. One of F' and F'' is F_1, of course, while the other graph is necessarily obtained by deleting a non-cut-vertex from a component F_i, $2 \leqslant i \leqslant k$. Hence, we need only remove a vertex which is not a cut-vertex from F_2, say, to produce the graph among F' and F'' which is not F_1. The other graph is then F_1.

Case 3. *At least one component among the F_i, $2 \leqslant i \leqslant k$, has order $m+1$ and all others have order $m+2$.* Consider all subgraphs $G-v_i$ with k components having a component of order m. In each such subgraph $G-v_i$, every component having order greater than m is a component of G. Thus, in this case, a graph H is a component of G if and only if H has order exceeding m and is a component of a subgraph $G-v_i$ with k components, one of which has order m. If each such subgraph $G-v_i$ has $k-1$ components isomorphic to H, then it follows immediately that all k components of G are isomorphic to H. If not all components of G are isomorphic to H, then one only needs to observe that number of components of G isomorphic to H is the maximum number of components isomorphic to H among the subgraphs $G-v_i$ with k components, one of which has order m, except if G has components of order $m+2$, H is a component of order $m+1$, and every component of order $m+1$ in each such $G-v_i$ is isomorphic to H. In this latter situation the number of components of G isomorphic to H is one greater than the afore-mentioned maximum.

This completes the proof.

One final remark might be in order here. By the *complement* $\bar{G}$ of a graph G is meant that graph having the same vertex set as G and in which two vertices u and v are adjacent if and only if u and v are not adjacent in G. Let $V(G) = \{v_1, v_2, \ldots, v_p\}, p \geqslant 3$, and suppose we are given the subgraphs $G-v_i$, $i = 1, 2, \ldots, p$. If G is connected, we may still be able to employ the theorem to reconstruct G. First, we determine the graphs $\overline{G-v_i}$, $i = 1, 2, \ldots, p$. It follows directly that $\overline{G-v_i} = \bar{G}-v_i$; hence, if we find that $\bar{G}$ is disconnected, we can reconstruct $\bar{G}$ and then determine G immediately. Of course, this procedure offers no technique to handle a connected graph whose complement is also connected. These remarks have also been observed by Harary and Kelly (see [3] and [4]).

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