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SEMI-INVARIANT SUBMANIFOLDS OF A SASAKIAN SPACE FORM

BY

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0. Introduction. Semi-invariant submanifolds of a Sasakian manifold have been introduced by the authors of the present paper in [1].

The main purpose of the paper is to obtain new results on the geometry of semi-invariant submanifolds in a Sasakian manifold.

First, in Section 1, we recall some fundamental results from [1], which are used in the next sections.

In Section 2 we study the geometry of leaves of distributions which are involved in the definition of a semi-invariant submanifold.

In Section 3 we obtain results on semi-invariant submanifolds of a Sasakian space form. We deal with some special classes of semi-invariant submanifolds with respect to their second fundamental form.

Finally, in Section 4, we study the natural f-structure on a semi-invariant submanifold M and obtain theorems on decomposition for M.

1. Semi-invariant submanifolds of a Sasakian manifold. Let $\tilde{M}$ be a $(2n+1)$-dimensional almost contact metric manifold with (F, ξ, η, g) as the almost contact metric structure, where F is a tensor field of type $(1, 1)$, ξ is a vector field, η is a 1-form, and g is a Riemannian metric on $\tilde{M}$. These tensor fields satisfy

$$F^2 = -I+\eta\otimes\xi, \quad F(\xi) = 0, \quad \eta\cdot F = 0, \quad \eta(\xi) = 1, \tag{1.1}$$

$$g(FX, FY) = g(X, Y)-\eta(X)\eta(Y) \tag{1.2}$$

for all vector fields X, Y tangent to $\tilde{M}$, where I denotes the identity morphism of the tangent bundle $T\tilde{M}$.

All the manifolds and morphisms considered in this paper are assumed to be differentiable of class C^∞.

The Levi-Cività connection on $\tilde{M}$ is denoted by $\tilde{\nabla}$. It is known that $\tilde{M}$ is a Sasakian manifold if and only if ([2], p. 73)

$$(\tilde{\nabla}_X F) Y = g(X, Y)\xi-\eta(Y)X \tag{1.3}$$

for all X, Y tangent to $\tilde{M}$. Also, from (1.3) we obtain

$$\tilde{\nabla}_X \xi = -FX. \tag{1.4}$$

Let M be an m-dimensional Riemannian manifold isometrically immersed in a Sasakian manifold $\tilde{M}$. Denote by TM the tangent bundle of M and by $TM^\perp$ the normal bundle to M. We assume that the structure vector field ξ of $\tilde{M}$ is tangent to M and denote by $\{\xi\}$ the 1-dimensional distribution defined by ξ on M.

Definition. The submanifold M of the Sasakian manifold $\tilde{M}$ is called *semi-invariant* if it is endowed with the pair of distributions $(D, D^\perp)$ satisfying the following conditions:

(i) $TM = D \oplus D^\perp \oplus \{\xi\}$, and D, $D^\perp$, $\{\xi\}$ are mutually orthogonal to each other;

(ii) the distribution D is *invariant* by F, that is, $F(D_x) = D_x$ for each $x \in M$;

(iii) the distribution $D^\perp$ is *anti-invariant* by F, that is, $F(D_x^\perp) \subset T_x M^\perp$ for each $x \in M$ (see [1]).

D and $D^\perp$ are called the *invariant distribution* and the *anti-invariant distribution* on M, respectively. Suppose the dimension of D_x (respectively, $D_x^\perp$) is $2p$ (respectively, q). Then it is easily seen that for $p = 0$ (respectively, $q = 0$) the semi-invariant submanifold M becomes an anti-invariant submanifold [4] (respectively, an invariant submanifold [3]).

A *generic semi-invariant submanifold* is characterized by the condition $q = \dim T_x M^\perp$ for any $x \in M$. Fundamental results on the geometry of generic submanifolds have been obtained by Yano and Kon in [6]. The semi-invariant submanifold M is called a *proper semi-invariant submanifold* if it is neither an invariant submanifold nor an anti-invariant submanifold. As we have seen in [1], each hypersurface of $\tilde{M}$ which is tangent to ξ is a typical example of proper semi-invariant submanifold.

Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. We denote by the same symbol g both metrics on M and $\tilde{M}$. The projection morphisms of TM to D and $D^\perp$ are denoted by P and Q, respectively. If H is a vector bundle over M, then we denote by $\Gamma(H)$ the module of all differentiable sections of H.

Using this notation we have

$$X = PX + QX + \eta(X)\xi \tag{1.5}$$

for all $X \in \Gamma(TM)$ and

$$FN = BN + CN \tag{1.6}$$

for all $N \in \Gamma(TM^\perp)$, where $BN \in \Gamma(D^\perp)$ and $CN \in \Gamma(TM^\perp)$.

We denote by ∇ the Levi-Cività connection on M and by $\nabla^\perp$ the linear connection induced by $\tilde{\nabla}$ on the normal bundle $TM^\perp$. Then the equations of

Gauss and Weingarten take the form

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1.7}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{1.8}$$

respectively, for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$, where h is the second fundamental form of M and A_N is the fundamental tensor of Weingarten with respect to the normal section N. Moreover, we have

$$g(h(X, Y), N) = g(A_N X, Y). \tag{1.9}$$

We need the following results obtained in [1].

LEMMA 1.1. *Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. Then*

$$P\nabla_X FPY - PA_{FQY} X = FP\nabla_X Y - \eta(Y) PX, \tag{1.10}$$

$$Q\nabla_X FPY - QA_{FQY} X = Bh(X, Y) - \eta(Y) QX, \tag{1.11}$$

$$\eta(\nabla_X FPY - A_{FQY} X) = g(FX, FY), \tag{1.12}$$

$$h(X, FPY) + \nabla^{\perp}_X FQY = Ch(X, Y) + FQ\nabla_X Y \tag{1.13}$$

for all vector fields $X, Y \in \Gamma(TM)$.

LEMMA 1.2. *Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. Then for any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$ we have*

$$FPA_N X = PA_{CN} X - P\nabla_X BN, \tag{1.14}$$

$$B\nabla^{\perp}_X N = Q\nabla_X BN - QA_{CN} X, \tag{1.15}$$

$$h(X, BN) + \nabla^{\perp}_X CN + FQA_N X = C\nabla^{\perp}_X N, \tag{1.16}$$

$$\eta(\nabla_X BN - A_{CN} X) = 0. \tag{1.17}$$

LEMMA 1.3. *Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. Then*

$$h(X, \xi) = 0, \quad \nabla_X \xi = -FX \quad \textit{for any } X \in \Gamma(D), \tag{1.18}$$

$$h(Y, \xi) = -FY, \quad \nabla_Y \xi = 0 \quad \textit{for any } Y \in \Gamma(D^{\perp}), \tag{1.19}$$

$$A_{FU} V = A_{FV} U \quad \textit{for all } U, V \in \Gamma(D^{\perp}), \tag{1.20}$$

$$h(\xi, \xi) = 0, \quad \nabla_{\xi} \xi = 0. \tag{1.21}$$

THEOREM 1.1. *Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. Then*

(i) *the distributions $D^{\perp}$ and $D^{\perp} \oplus \{\xi\}$ are always involutive;*

(ii) *the distributions D and $D \oplus D^{\perp}$ are never involutive;*

(iii) *the distribution* $D \oplus \{\xi\}$ *is involutive if and only if the second fundamental form of* M *satisfies*

$$h(X, FY) = h(FX, Y) \tag{1.22}$$

for all $X, Y \in \Gamma(D)$.

The submanifold M is called *totally geodesic* if h vanishes identically on M. From (1.19) we obtain

LEMMA 1.4. (a) *There exist no totally geodesic proper semi-invariant submanifolds in a Sasakian manifold.*

(b) *If* M *is a totally geodesic semi-invariant submanifold of a Sasakian manifold, then it has to be an invariant submanifold.*

Now we introduce a weaker condition on the second fundamental form h. We say that M is $(D, D^{\perp})$*-geodesic* if $h(X, Y) = 0$ for all $X \in \Gamma(D)$ and $Y \in \Gamma(D^{\perp})$. Then, using (1.9) and (1.18), we obtain

LEMMA 1.5. *Let* M *be a semi-invariant submanifold of a Sasakian manifold* $\tilde{M}$. *Then* M *is* $(D, D^{\perp})$*-geodesic if and only if one of the following conditions is satisfied*:

(1) $A_N X \in \Gamma(D)$ *for any* $X \in \Gamma(D)$ *and* $N \in \Gamma(TM^{\perp})$;

(2) $A_N Y \in \Gamma(D^{\perp} \oplus \{\xi\})$ *for any* $Y \in \Gamma(D^{\perp} \oplus \{\xi\})$ *and* $N \in \Gamma(TM^{\perp})$.

2. Geometry of leaves on a semi-invariant submanifold in a Sasakian manifold. Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. The purpose of this section is to study the geometry of leaves of distributions involved in the definition of M.

LEMMA 2.1. *Let* M *be a semi-invariant submanifold of a Sasakian manifold* $\tilde{M}$. *Then*

$$h(X, FY) = h(FX, Y) \quad \textit{for all } X, Y \in \Gamma(D) \tag{2.1}$$

if and only if

$$g(h(X, FY), FZ) = g(h(FX, Y), FZ) \tag{2.2}$$

$$\textit{for all } X, Y \in \Gamma(D) \textit{ and } Z \in \Gamma(D^{\perp}).$$

Proof. It is easily seen that we have only to prove that (2.2) implies

$$Ch(X, FY) = Ch(FX, Y) \quad \text{for all } X, Y \in \Gamma(D). \tag{2.3}$$

We take $N \in \Gamma(\tilde{D})$, where $\tilde{D}$ is the complementary orthogonal subbundle to $F(D^{\perp})$ in $TM^{\perp}$. Then, using (1.2), (1.9), and (1.14), we obtain

$$\begin{aligned} (2.4) \quad & g(h(X, FY), N) \\ &= g(A_N X, FY) = g(PA_N X, FY) = -g(FPA_{FN} X + P\nabla_X BFN, FY) \\ &= -g(FPA_{FN} X, FY) = -g(A_{FN} X, Y) = -g(h(X, Y), FN). \end{aligned}$$

Thus, (2.3) follows from (2.4) because h is a symmetric tensor field on M.

From (iii) of Theorem 1.1 and Lemma 2.1 we get

COROLLARY 2.1. *Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. If $A_{FZ}X \in \Gamma(D^{\perp})$ for all $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, then the distribution $D \oplus \{\xi\}$ is involutive.*

The semi-invariant submanifold M is called *$(D \oplus \{\xi\})$-geodesic* if

$$h(X, Y) = 0 \quad \text{for all } X, Y \in \Gamma(D \oplus \{\xi\}). \tag{2.5}$$

By (1.18) and (1.21) we see that M is $(D \oplus \{\xi\})$-geodesic if and only if (2.5) holds for all $X, Y \in \Gamma(D)$.

THEOREM 2.1. *Let M be a $(D \oplus \{\xi\})$-geodesic semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. Then each leaf of the distribution $D \oplus \{\xi\}$ is totally geodesic immersed in $\tilde{M}$.*

Proof. First, using (1.3) and (1.7)–(1.9) we obtain

$$g(\nabla_X Z, FY) = g(h(X, Y), FZ) \tag{2.6}$$

$$\text{for all } X, Y \in \Gamma(D \oplus \{\xi\}) \text{ and } Z \in \Gamma(D^{\perp}).$$

Next, since M is $(D \oplus \{\xi\})$-geodesic, using (1.22) and (2.5) we infer that $D \oplus \{\xi\}$ is involutive. Let M^* be a leaf of the distribution $D \oplus \{\xi\}$. Then, from (2.6) we get

$$g(A_Z^* X, Y) = -g(h(X, FY), FZ) = 0 \tag{2.7}$$

for all $X \in \Gamma(TM^*)$, $Y \in \Gamma(D)$, and for Z normal to M^* but tangent to M, where A_Z^* is the Weingarten tensor of M^* with respect to Z.

On the other hand, using the Weingarten equation for the immersion of M^* in M and (1.18) we have

$$g(A_Z^* X, \xi) = -g(\nabla_X Z, \xi) = g(Z, \nabla_X \xi) = -g(Z, FX) = 0 \tag{2.8}$$

for all X and Z from (2.7). Thus, by (2.7) and (2.8), M^* is totally geodesic immersed in M. Finally, by (2.5), M^* is totally geodesic immersed in $\tilde{M}$.

As we have seen in Theorem 1.1 the distribution $D^{\perp}$ is always involutive. We denote by $M^{\perp}$ a leaf of the distribution $D^{\perp}$.

THEOREM 2.2. *The leaf $M^{\perp}$ is totally geodesic immersed in the semi-invariant submanifold M if and only if*

$$h(X, Y) \in \Gamma(\tilde{D}) \quad \textit{for all } X \in \Gamma(D^{\perp}) \textit{ and } Y \in \Gamma(D). \tag{2.9}$$

Proof. Using (1.9) and (1.10) we obtain

$$g(FP\nabla_X Z, Y) = -g(h(X, Y), FZ) \tag{2.10}$$

for all $X, Z \in \Gamma(D^{\perp})$ and $Y \in \Gamma(D)$. We denote by ∇' the linear connection induced by ∇ on $M^{\perp}$ and by h' the second fundamental form of the immersion of $M^\perp$ in M. Hence the Gauss equation is of the form

$$\nabla_X Z = \nabla'_X Z + h'(X, Z) \tag{2.11}$$

for all X, Z tangent to $M^\perp$. Then from (2.9)–(2.11) we obtain

$$g(h'(X, Z), FY) = g(h(X, Y), FZ) = 0 \tag{2.12}$$

for all X, Z tangent to $M^\perp$ and $Y \in \Gamma(D)$.

On the other hand, by (1.17) we have $\eta(\nabla_X Z) = 0$, which together with (2.11) implies

$$\eta(h'(X, Z)) = 0 \tag{2.13}$$

for all X, Z tangent to $M^\perp$. Finally, the theorem follows from (2.12) and (2.13).

The semi-invariant submanifold M is called *$D^\perp$-geodesic* if

$$h(X, Y) = 0 \quad \text{for all } X, Y \in \Gamma(D^\perp). \tag{2.14}$$

Now, using Theorem 2.2 we prove

THEOREM 2.3. *Let M be a $(D, D^\perp)$-geodesic semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. Then*

(1) *each leaf of the anti-invariant distribution is totally geodesic immersed in M;*

(2) *each leaf of the anti-invariant distribution is totally geodesic immersed in the Sasakian manifold $\tilde{M}$ if and only if M is $D^\perp$-geodesic.*

Proof. Let $M^\perp$ be an arbitrary leaf of the anti-invariant distribution. We denote by h' and $\tilde{h}$ the second fundamental forms of $M^\perp$ in M and $\tilde{M}$, respectively. Then we have

$$\tilde{h}(X, Y) = h(X, Y) + h'(X, Y) \tag{2.15}$$

for all X, Y tangent to $M^\perp$. The assertion (1) follows from Theorem 2.2, and the assertion (2) is implied by (1) and (2.15).

From Theorems 2.2 and 2.3 we obtain

COROLLARY 2.2. *A generic semi-invariant submanifold M of a Sasakian manifold $\tilde{M}$ is $(D, D^\perp)$-geodesic if and only if each leaf of the anti-invariant distribution is totally geodesic immersed in M.*

Moreover, concerning the immersion of each leaf of $D^\perp$ in $\tilde{M}$ we have

THEOREM 2.4. *Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. Then each leaf of $D^\perp$ is totally geodesic immersed in $\tilde{M}$ if and only if the following conditions are fulfilled:*

(1) $\nabla^\perp_X FY \in \Gamma(F(D^\perp))$ *for all* $X, Y \in \Gamma(D^\perp)$;

(2) $h(X, Z) \in \Gamma(\tilde{D})$ *for all* $X \in \Gamma(D^\perp)$ *and* $Z \in \Gamma(D \oplus D^\perp)$.

Proof. For all X, $Y \in \Gamma(D^{\perp})$, adding (1.11) and (1.13), we obtain

$$h(X, Y) = FQ(A_{FY} X) - F(\nabla_X^{\perp} FY) - Q(\nabla_X Y) \tag{2.16}$$

since $\eta(h(X, Y)) = \eta(Q\nabla_X Y) = 0$. Next, (1.6) together with (2.16) implies

$$h(X, Y) = FQ(A_{FY} X) - C(\nabla_X^{\perp} FY) \quad \text{for all } X, Y \in \Gamma(D^{\perp}). \tag{2.17}$$

Suppose each leaf of $D^{\perp}$ is totally geodesic immersed in the Sasakian manifold $\tilde{M}$. Then, by (2.15) and Theorem 2.2, we obtain condition (2). Moreover, (2.17) implies $C(\nabla_X^{\perp} FY) = 0$, which is equivalent to condition (1).

Conversely, suppose conditions (1) and (2) are satisfied. Then, from (2.17) we get $h(X, Y) = 0$ for all X, $Y \in \Gamma(D^{\perp})$. On the other hand, from condition (2) and Theorem 2.2 we infer that any leaf of $D^{\perp}$ is totally geodesic immersed in M. Finally, by (2.15), any leaf of $D^{\perp}$ is totally geodesic immersed in the Sasakian manifold $\tilde{M}$.

THEOREM 2.5. *Let M be a generic semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. Then each leaf of the anti-invariant distribution is totally geodesic in $\tilde{M}$ if and only if M is $(D, D^{\perp})$-geodesic and $D^{\perp}$-geodesic immersed in $\tilde{M}$.*

The theorem follows from Theorem 2.4.

3. Semi-invariant submanifolds of a Sasakian space form. Suppose that $\tilde{M}(c)$ is a Sasakian space form of constant F-sectional curvature c and that M is a semi-invariant submanifold of $\tilde{M}(c)$. The curvature tensor $\tilde{R}$ of $\tilde{M}(c)$ is given by

$$\begin{aligned} \tilde{R}(X, Y)Z = \frac{c+3}{4}\{g(Y, Z)X - g(X, Z)Y\} + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \\ -\eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi + \\ + g(Z, FY)FX - g(Z, FX)FY + 2g(X, FY)FZ\} \end{aligned} \tag{3.1}$$

for all X, Y, Z tangent to $\tilde{M}(c)$.

Then the Codazzi equation is of the form

$$\begin{aligned} &(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) \\ &= \frac{c-1}{4}\{g(Z, FPY)FQX - g(Z, FPX)FQY + 2g(X, FPY)FQZ\} \end{aligned} \tag{3.2}$$

for all X, Y, Z tangent to M, where $\nabla_X h$ is defined by

$$(\nabla_X h)(Y, Z) = \nabla_X^{\perp}(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{3.3}$$

We say that the second fundamental form h of M satisfies the *classical equation of Codazzi* if

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z) \tag{3.4}$$

for all X, Y, $Z \in \Gamma(TM)$. Then we can state

THEOREM 3.1. *Let M be a semi-invariant submanifold of a Sasakian space form $\tilde{M}(c)$ with $c \neq 1$. If the second fundamental form of M satisfies the classical equation of Codazzi, then M is either an anti-invariant submanifold or an invariant submanifold.*

Proof. Suppose M is not an anti-invariant submanifold of $\tilde{M}(c)$. Then in (3.2) we take a unit vector field $X \in \Gamma(D)$ and $Y = FX$. Using (3.4) and taking into account that $c \neq 1$ we obtain $FQZ = 0$ for each $Z \in \Gamma(TM)$. This means that M is an invariant submanifold.

The semi-invariant submanifold M of a Sasakian manifold $\tilde{M}$ is called *$(D, D^{\perp})$-foliate* if it is $(D, D^{\perp})$-geodesic and the distribution $D \oplus \{\xi\}$ is involutive.

THEOREM 3.2. *A semi-invariant submanifold M of the Sasakian manifold $\tilde{M}$ is $(D, D^{\perp})$-foliate if and only if*

$$A_N FP + FPA_N = 0 \tag{3.5}$$

for all $N \in \Gamma(TM^{\perp})$.

Proof. By means of (iii) of Theorem 1.1 we see that M is $(D, D^{\perp})$-foliate if and only if

$$g\big(h(X, FPY), N\big) = g\big(h(FPX, Y), N\big) \tag{3.6}$$

for all $X, Y \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$. Then, using (1.9) we obtain the equivalence of (3.6) and (3.5).

THEOREM 3.3. *Let M be a $(D, D^{\perp})$-foliate proper semi-invariant submanifold of a Sasakian space form $\tilde{M}(c)$. Then $c \leqslant -3$.*

Proof. We take $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$ such that $Z = FN$ for a certain section $N \in \Gamma\big(F(D^{\perp})\big)$. Then, by (1.14), (1.15), and (1.17) we obtain

$$\nabla_Y Z = B\nabla_Y^{\perp} N - FPA_N Y \tag{3.7}$$

since $CN = 0$. Next, by (1.13), (1.19), (3.3), (3.6), and (3.7), we have

$$\begin{aligned}(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) \\ = h(Y, FPA_N X) - h(X, FPA_N Y) + \eta(\nabla_X Y)FZ - \eta(\nabla_Y X)FZ\end{aligned} \tag{3.8}$$

for all $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$ such that $Z = FN$, where $N \in \Gamma\big(F(D^{\perp})\big)$. Substituting FY for X in (3.8) and using (3.2), (3.5), (1.2), and (1.12) we obtain

$$2h(FY, A_N FY) + 2g(FY, FY)N = \frac{1-c}{2}\, g(FY, FY)N. \tag{3.9}$$

Finally, from (3.9) we get

$$0 \leqslant g(A_N FY, A_N FY) = -\frac{c+3}{4}\, g(FY, FY)g(N, N), \tag{3.10}$$

which implies $c \leqslant -3$.

From Theorem 3.3 we obtain

COROLLARY 3.1. *Let M be a $(D, D^{\perp})$-foliate semi-invariant submanifold of $\tilde{M}(c)$. If $c > -3$, then M is either an anti-invariant submanifold or an invariant submanifold.*

4. f-structure on a semi-invariant submanifold. Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. We put $\varphi = F \circ P$ and, using (1.1), we obtain $\varphi^3 + \varphi = 0$, that is, φ is an f-structure on M (see [5]). The purpose of this section is to study the fundamental properties of the f-structure φ. First, by direct computation using (1.2), (1.3), (1.5), and (1.10)–(1.12) we obtain

LEMMA 4.1. *Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. Then*

$$(4.1) \qquad Bh(X, Y) = (\nabla_X \varphi) Y - A_{FQY} X - (\tilde{\nabla}_X F) Y$$

for all X, $Y \in \Gamma(TM)$.

The f-structure φ is said to be *parallel* if $\nabla_X \varphi = 0$ for any $X \in \Gamma(TM)$.

THEOREM 4.1. *Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. If the f-structure φ is parallel, then M is an anti-invariant submanifold of $\tilde{M}$.*

Proof. We take $Y \in \Gamma(D)$ in (4.1) and using (1.3) we obtain $Bh(X, Y) + g(X, Y)\xi = 0$, which implies $g(X, Y) = 0$ for all $X \in \Gamma(TM)$. This is possible only when M is an anti-invariant submanifold.

Remark. From the proof of Theorem 4.1 we see that its assertion follows even when $(\nabla_X \varphi) Y = 0$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$.

It is interesting to find weaker conditions than that of φ being parallel and obtain theorems of decomposition for proper semi-invariant submanifolds. The remaining part of the paper deals with this problem.

Definition. The f-structure φ is said to be *η-parallel* if for all X, $Y \in \Gamma(TM)$ we have

$$(4.2) \qquad (\nabla_X \varphi) Y = g(PX, PY)\xi - \eta(Y) PX.$$

LEMMA 4.2. *Let M be a semi-invariant submanifold of a Sasakian manifold $\tilde{M}$. Then the f-structure φ is η-parallel if and only if*

$$(4.3) \qquad Bh(X, Y) = \eta(Y) QX - QA_{FQY} X$$

for all X, $Y \in \Gamma(TM)$.

Proof. Using (1.3), (1.18) and (1.19) in (4.1) we obtain

$$(4.4) \qquad Bh(X, Y) = \{(\nabla_X \varphi) Y - g(PX, PY)\xi + \eta(Y) PX\} + \{\eta(Y) QX - QA_{FQY} X - PA_{FQY} X\}.$$

On the other hand, (4.3) together with (1.9) implies

$$(4.5) \qquad PA_{FQY} X = 0 \quad \text{for all } X, Y \in \Gamma(TM).$$

Finally, the lemma follows from (4.4) and (4.5).

THEOREM 4.2. *Let M be an m-dimensional generic semi-invariant submanifold of a $(2n+1)$-dimensional Sasakian manifold $\tilde{M}$. If the f-structure φ is η-parallel, then M is locally the Riemannian direct product $M_1 \times M_2$, where M_1 is a $(2m-2n-1)$-dimensional totally geodesic invariant submanifold of $\tilde{M}$ and M_2 is a $(2n-m+1)$-dimensional anti-invariant submanifold such that the structure vector field ξ is normal to M_2.*

Proof. Since M is a generic semi-invariant submanifold, we have $F = B$ on $TM^{\perp}$. Then, from (4.3) we get

$$h(X, Y) = 0 \quad \text{for all } X \in \Gamma(TM) \text{ and } Y \in \Gamma(D). \tag{4.6}$$

By (4.6) and (1.13) we obtain

$$\nabla_X^{\perp} FQY - FQ\nabla_X Y = 0 \quad \text{for all } X, Y \in \Gamma(TM). \tag{4.7}$$

Now, we take $Y \in \Gamma(D \oplus \{\xi\})$. Then (4.7) implies $\nabla_X Y \in \Gamma(D \oplus \{\xi\})$ for each $X \in \Gamma(TM)$. Consequently, the distribution $D \oplus \{\xi\}$ is parallel. For any $Y \in \Gamma(D^{\perp})$ from (4.2) we obtain $FP\nabla_X Y = 0$, which implies $\nabla_X Y \in \Gamma(D^{\perp} \oplus \{\xi\})$ for any $X \in \Gamma(TM)$.

On the other hand, from (1.17) we get $\eta(\nabla_X Y) = 0$ for all $X \in \Gamma(TM)$ and $Y \in \Gamma(D^{\perp})$. Hence $\nabla_X Y \in \Gamma(D^{\perp})$ for all $X \in \Gamma(TM)$ and $Y \in \Gamma(D^{\perp})$, which means that the distribution $D^{\perp}$ is also parallel. Therefore, M is locally the Riemannian direct product $M_1 \times M_2$, where M_1 is a leaf of the distribution $D \oplus \{\xi\}$ and M_2 is a leaf of $D^{\perp}$. Of course, M_1 is an invariant submanifold of $\tilde{M}$ and M_2 is an anti-invariant submanifold such that ξ is normal to M_2. Moreover, by (4.6) the conditions of Theorem 2.1 are satisfied. Hence M_1 is totally geodesic immersed in $\tilde{M}$. The proof is complete.

THEOREM 4.3. *Let M be an m-dimensional generic semi-invariant submanifold of a $(2n+1)$-dimensional Sasakian space form $\tilde{M}(c)$ with $c \neq -3$. Suppose the f-structure φ of M is η-parallel. Then M is an anti-invariant submanifold of $\tilde{M}(c)$.*

Proof. By a direct computation, using (4.6) we obtain $(\nabla_X h)(Y, Z) = -h(Y, \nabla_X Z)$ for all $X, Y \in \Gamma(TM)$ and $Z \in \Gamma(D)$. Then from (3.2) we get

$$\frac{c-1}{4}\{g(Z, FPY)FQX - g(Z, FPX)FQY\} = h(X, \nabla_Y Z) - h(Y, \nabla_X Z) \tag{4.8}$$

for all $X, Y \in \Gamma(TM)$ and $Z \in \Gamma(D)$. In (4.8) we take $Y \in \Gamma(D)$, $Z = FY$, and by (4.6) and Lemma 1.3 we obtain

$$\frac{c-1}{4} g(FY, FY)FQX = -\eta(\nabla_Y FY)FQX \tag{4.9}$$

for all $X \in \Gamma(TM)$.

Using (1.12) together with (4.9) we have

$$\frac{c+3}{4}\, g(FY, FY)FQX = 0 \tag{4.10}$$

for all $Y \in \Gamma(D)$ and $X \in \Gamma(TM)$. Since M is a generic semi-invariant submanifold and $c \neq -3$, from (4.10) we obtain $FY = 0$ for all $Y \in \Gamma(D)$. Thus M is an anti-invariant submanifold. The proof is complete.

By $R^{2n+1}(-3)$ we mean the Sasakian space form with constant F-sectional curvature $c = -3$ and with standard Sasakian structure in the Euclidean space R^{2n+1} (see [2], p. 99).

THEOREM 4.4. *Let M be a generic $(D, D^{\perp})$-foliate proper semi-invariant submanifold of the Sasakian space form $R^{2n+1}(-3)$. Then M is locally the Riemannian product $M_1 \times M_2$, where M_1 is a $(2m-2n-1)$-dimensional totally geodesic invariant submanifold of $R^{2n+1}(-3)$ and M_2 is a $(2n-m+1)$-dimensional anti-invariant submanifold such that the structure vector field ξ is normal to M_2.*

Proof. From (3.10) we get $A_N X = 0$ for all $X \in \Gamma(D)$ and $N \in \Gamma(TM^{\perp})$. Thus (4.3) is satisfied for any $X \in \Gamma(D)$ and $Y \in \Gamma(TM)$. Using (i) of Theorem 1.1 and (1.19) we obtain

$$g(\nabla_{\xi} FPY, Z) = 0 \tag{4.11}$$

for any $Y \in \Gamma(TM)$ and $Z \in \Gamma(D^{\perp})$. Now, (1.11) together with (4.11) implies (4.3) for all $Y \in \Gamma(TM)$ and $X = \xi$. Next, we take $X \in \Gamma(D^{\perp})$ and $Y \in \Gamma(D^{\perp} \oplus \{\xi\})$ in (1.11) and obtain (4.3). Finally, since M is $(D, D^{\perp})$-geodesic, we obtain (4.3) for $X \in \Gamma(D^{\perp})$ and $Y \in \Gamma(D)$. Thus we infer that (4.3) holds for all $X, Y \in \Gamma(TM)$. Hence, by Lemma 4.2, the f-structure φ on M is η-parallel and, by Theorem 4.2, the proof is complete.

The second fundamental form h of a semi-invariant submanifold M is said to be *parallel* if $\nabla_X h = 0$ for all $X \in \Gamma(TM)$. From Theorem 3.1 we obtain

COROLLARY 4.1. *Let M be a generic semi-invariant submanifold of a Sasakian space form $\tilde{M}(c)$ with $c \neq 1$. If the second fundamental form of M is parallel, then M is an anti-invariant submanifold of $\tilde{M}(c)$.*

By $S^{2n+1}(1)$ we mean a $(2n+1)$-dimensional sphere endowed with the natural Sasakian structure of constant F-sectional curvature $c = 1$ (see [2], p. 99).

THEOREM 4.5. *Let M be a generic semi-invariant submanifold of the sphere $S^{2n+1}(1)$. If the second fundamental form h of M is parallel and the distribution $D \oplus \{\xi\}$ is involutive, then M is an anti-invariant submanifold of $S^{2n+1}(1)$.*

Proof. Since M is a generic semi-invariant submanifold, we have $C = 0$. Suppose $\nabla_X h = 0$ for all $X \in \Gamma(TM)$, that is,

$$\nabla_X^{\perp}(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) = 0 \tag{4.12}$$

for all X, Y, $Z \in \Gamma(TM)$. In (4.12) we take $X \in \Gamma(D^{\perp})$ and $Z = \xi$. Then, using (1.19) and (1.13) we obtain

$$h(X, FPY) = 0 \tag{4.13}$$

for all $X \in \Gamma(D^{\perp})$ and $Y \in \Gamma(TM)$. From (4.13) it follows that M is a $(D, D^{\perp})$-geodesic semi-invariant submanifold of $S^{2n+1}(1)$. Since the distribution $D \oplus \{\xi\}$ is involutive, M is a $(D, D^{\perp})$-foliate semi-invariant submanifold. Taking into account that $c = 1$ and M is generic, we see that the assertion of theorem follows from Corollary 3.1.

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