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A TOPOLOGICAL REPRESENTATION OF POST ALGEBRAS AND FREE POST ALGEBRAS

BY

ZYGMUNT SALONI (WARSZAWA)

In this paper we consider properties of Stone spaces of Post algebras of order n treated as distributive lattices and we construct free Post algebras of order n. Prerequisities for a construction of free Post algebras with a finite number of generators were given by Traczyk [10].

If $\boldsymbol{A}$ is a distributive lattice (we do not distinguish between symbols for an algebra and for the set of its elements), we denote its Stone space by A^* and the algebra of open subsets of this space, isomorphic to $\boldsymbol{A}$, by $\boldsymbol{A}^*$. If $\boldsymbol{A}$ is a distributive lattice, then A^* is the set of all its prime filters (ideals) with topology given by the Stone base, i.e. the set of all $h(a) = \{\nabla\colon a \in \nabla\}$ $(\{\Delta\colon a \notin \Delta\})$ for $a \in \boldsymbol{A}$.

We consider a Post algebra of order n as an abstract algebra

$$\boldsymbol{P} = (\boldsymbol{P}, \rightarrow, \cup, \cap, \neg, D_1, \ldots, D_{n-1}, e_0, \ldots, e_{n-1}).$$

Axioms for this system have been given by Rousseau [5], and used by other authors, e.g. [3] and [6]. Equivalent systems of axioms, without $\rightarrow$ and $\neg$ which can be defined by other operators, are given in other papers, e.g. [2], [9] and [11]. Introduction of operators $\rightarrow$ and $\neg$ is motivated by logical applications of Post algebras and allows to consider them as a class of pseudo-Boolean (Heyting) algebras.

Every Post algebra of order n is a coproduct of a Boolean algebra $\boldsymbol{B}$ and an n-element chain $\boldsymbol{N}$ (cf. [5] and [8]) and, therefore, we shall occasionally write $\boldsymbol{P} = \boldsymbol{B} * \boldsymbol{N}$.

A Post homomorphism $\varphi\colon \boldsymbol{P} \rightarrow \boldsymbol{P}'$, where $\boldsymbol{P} = \boldsymbol{B} * \boldsymbol{N}$ and $\boldsymbol{P}' = \boldsymbol{B}' * \boldsymbol{N}'$ are Post algebras of order n and n', respectively, introduced by Traczyk [9] and Dwinger [1], is a function from $\boldsymbol{P}$ into $\boldsymbol{P}'$ satisfying the following conditions:

a. φ is a lattice homomorphism;

b. $\varphi(e_j) \in \boldsymbol{N}'$ for $j = 0, \ldots, n-1$.

In Section 3 we restrict considerations to a special case, which seems interesting from an algebraical and logical point of view, where $\boldsymbol{P}$ and

$\boldsymbol{P}'$ are of the same order n and $\varphi(e_j) = e_j$ for every $j = 1, \ldots, n-1$. Every such homomorphism is determined by a homomorphism of the Boolean algebra $\boldsymbol{B}$ only, and not by a homomorphism of the chain (which, in this case, must be an isomorphism). If we treat the chain in a Post algebra as the set of constants of an abstract algebra, then only such homomorphisms are proper in the algebraical sense.

1. Stone spaces for Post algebras. Let $\boldsymbol{N} = \{e_0, \ldots, e_{n-1}\}$ be an n-element chain that is a set ordered by $e_0 < \ldots < e_{n-1}$. It is well known that $\boldsymbol{N}$ is a Post algebra of order n with the following operations:

$$e_i \cup e_j = e_{\max(i,j)}, \qquad e_i \cap e_j = e_{\min(i,j)},$$

$$\neg e_j = \begin{cases} e_0 & \text{if } j > 0, \\ e_{n-1} & \text{if } j = 0, \end{cases} \qquad e_i \to e_j = \begin{cases} e_{n-1} & \text{if } i \leqslant j, \\ e_j & \text{if } i > j, \end{cases}$$

$$D_i(e_j) = \begin{cases} e_{n-1} & \text{if } i \leqslant j, \\ e_0 & \text{if } i > j. \end{cases}$$

The set $\{1, \ldots, n-1\}$ with the *order topology*, i.e. with the topology $\{\emptyset, [1,j]_{j=1}^{n}\}$, where $[1,j] = \{1, \ldots, j\}$, is the *Stone space* N^* for the distributive lattice $\boldsymbol{N}$. As it is easy to prove, the lattice $\boldsymbol{N}^*$ of open sets, considered with the additional operations

$$Y \to Z = \mathrm{Int}\big((N^* \setminus Y) \cup Z\big), \qquad \neg Y = Y \to \emptyset,$$

$$E_0 = \emptyset, \qquad E_j = [1,j] \text{ for } j = 1, \ldots, n-1,$$

$$D_j(Y) = \begin{cases} N^* = E_{n-1} & \text{if } j \in Y, \\ \emptyset = E_0 & \text{if } j \notin Y, \end{cases}$$

is a Post algebra isomorphic to $\boldsymbol{N}$.

Let B^* be a topological space and let $P^* = B^* \times N^*$. In the whole paper, $f\colon P^* \to N^*$ and $g\colon P^* \to B^*$ denote projections, and we use symbols f_t and g_t if $P_t^* = B_t^* \times N^*$.

A set-theoretic representation of Post algebras was given by Traczyk [12] and used by other authors, e.g. [1], [3] and [14]. A modification of the rather poor topology of these algebras (the space was not even a T_0-space and not all images of elements of algebra were open) was given by Speed [8]. The following lemma is a modification of his result applied to Post algebras of order n considered as pseudo-Boolean algebras:

LEMMA 1. *If $\boldsymbol{P}$ is a Post algebra of order n and $\boldsymbol{P} = \boldsymbol{B} * \boldsymbol{N}$, then $P^* = B^* \times N^*$. The space P^* is a compact T_0-space and, for $n > 2$, not a T_1-space. If $n \geqslant 2$, then P^* has the dimension $n-2$ in the sense that every open cover of P^* has an open refinement of order $n-2$. The Post algebra $\boldsymbol{P}$ is isomorphic to the algebra of open sets $\boldsymbol{P}^*$, where the operations $\cup$ and*

$\cap$ *are set-theoretic union and intersection, respectively, and the other operations are defined as follows:*

$$Y \to Z = \mathrm{Int}\big((P^* \setminus Y) \cup Z\big), \qquad \neg Y = Y \to \emptyset,$$

$$E_0 = \emptyset, \quad E_j = f^{-1}([1, j]) \ \textit{for } j = 1, \ldots, n-1,$$

$$D_j(Y) = g^{-1}g\big(Y \cap f^{-1}(j)\big) \quad \textit{for } j = 1, \ldots, n-1.$$

Moreover, $D_1(Y) = g^{-1}g(Y)$ *and* $\neg Y = g^{-1}\big(-g(Y)\big)$.

The first part of this lemma follows from the representation theorem for the coproduct of distributive lattices (cf. [7]) and was formulated in [8]. The topological properties of P^* follow from well-known topological facts. The isomorphism of $\boldsymbol{P}$ and $\boldsymbol{P}^*$, treated as distributive lattices, is evident. The proof of the definitions of the operators E_j and D_j can be given by an easy comparison with the respective operators in the Traczyk representation [12]. The topological representation of the operators $\to$ and $\neg$ is valid for every pseudo-Boolean algebra.

We propose the following definitions which seem to fit both the general theory of representations of distributive lattices and the special theory of Post algebras of order n:

Definition 1. A topological space is said to be a *Stone-Post space of order n* if it is a product of a totally disconnected compact Hausdorff space and an $(n-1)$-element chain with the order topology.

Definition 2. A *Post algebra of sets of order n* is a pseudo-field of open subsets of a Stone-Post space of order n, generated by the product subbase, enriched by the operations D_j (for $j = 1, \ldots, n-1$) and E_j (for $j = 0, \ldots, n-1$) defined as in Lemma 1.

COROLLARY 1. *If* $\boldsymbol{P}^*$ *is a Post algebra of sets of order n, i.e., a Post algebra of subsets of a topological space* $P^* = B^* \times N^*$, *then*

1. *The set of elements of* $\boldsymbol{P}^*$ *of the form* $g^{-1}(U)$ *for* $U \in \boldsymbol{B}^*$, *where* $\boldsymbol{B}^*$ *is the field of all open and closed subsets of* B^*, *is closed under all unary and binary operations and, if considered as an algebra with the operations* $\to$, $\cup$, $\cap$ *and* $\neg$, *is isomorphic to* $\boldsymbol{B}^*$. *The family of sets* $\{g^{-1}(U)\}_{U \in \boldsymbol{B}^*}$ *is the set of all open and closed subsets of* P^*.

2. *The set of elements of* $\boldsymbol{P}^*$ *of the form* $f^{-1}(E_j)$, *for* $E_j \in \boldsymbol{N}^*$, *is closed under all operations and isomorphic to* $\boldsymbol{N}^*$. *The sets* $f^{-1}(E_j)$ *for* $j = 1, \ldots, n-1$ *are open and dense in* P^*.

The following corollary is a reformulation of the Rousseau result [5] for Post algebras of sets. It binds the pseudo-Boolean operations $\to$ and $\neg$ with the usual set-theoretic operations on $g^{-1}(B^*)$ and with the constants.

COROLLARY 2. *In every Post algebra of sets of order n we have*

$$Y \to Z = \bigcup_{i=1}^{n-1} \Big(\bigcap_{j=1}^{i} \big(-D_j(Y) \cup D_j(Z) \big) \Big) \cap E_i, \qquad \neg Y = -D_1(Y).$$

In the above-given representation of Post algebras the theorem on preserving infinite operations (cf. [14]) is obvious.

2. Reduced product of Post algebras of sets of order n. Although in this section we restrict ourselves to Post algebras of sets, all conclusions will apply, by virtue of Lemma 1, to all Post algebras.

If $\prod_{t \in T} A_t$ denotes the Cartesian product of sets A_t, then p_{t_0} will denote the projection

$$\prod_{t \in T} A_t \to A_{t_0}.$$

Let, for $t \in T$, $P_t^* = B_t^* \times N^*$ be a Stone-Post space and $\boldsymbol{P}_t^*$ the Post algebra of its open subsets. Let

$$A^* = \prod_{t \in T} B_t^*$$

be the product of spaces B_t^*. The algebra $\boldsymbol{A}^*$ of its open subsets, generated by its product subbase, i.e., by the family of sets $p_{t_0}^{-1}(U_{t_0})$, for $t_0 \in T$ and U_{t_0} belonging to the Stone base of $B_{t_0}^*$, is a field of sets. Thus, the algebra of open subsets of $A^* \times N^*$, generated by its product subbase, is a Post algebra of sets of order n.

Now, let $\approx$ be the relation in the product $\prod_{t \in T} P_t^*$ defined as follows:

$$a \approx b \Leftrightarrow \Big(\big(\bigvee_{t \in T} g_t p_t(a) = g_t p_t(b) \big) \text{ and } \big(\min_{t \in T} f_t p_t(a) = \min_{t \in T} f_t p_t(b) \big) \Big).$$

It is obvious that $\approx$ is an equivalence relation.

LEMMA 2. *The spaces $\prod_{t \in T} P_t^* / \approx$ with the quotient topology and $A^* \times N^*$ are homeomorphic and the homeomorphism i is given by the formula*

$$i\big([(b_t, j_t)_{t \in T}]\big) = \big((b_t)_{t \in T}, \min j_t\big).$$

The lattice of open subsets of the space $\prod_{t \in T} P_t^* / \approx$, generated by counter-images of the product subbases of P_t^*, is isomorphic to the lattice of open subsets of the space $A^* \times N^*$, generated by counter-images of Stone bases of B_t^* and N^*. In the former, the operations $\to$, $\neg$, D_j (for $j = 1, \ldots, n-1$), and E_j (for $j = 0, \ldots, n-1$) are defined by

$$Y \to Z = i^{-1}\big(i(Y) \to i(Z)\big), \qquad \neg Y = i^{-1}\big(\neg i(Y)\big),$$
$$D_j(Y) = i^{-1}\big(D_j(i(Y))\big), \qquad E_j = i^{-1}(E_j),$$

and so it is a Post algebra of sets of order n.

Proof. The mapping i is one-to-one and onto, since every point of $\prod_{t\in T} P_t^* / \approx$, as well as of $A^* \times N^*$, is defined by t Boolean axes and one $(n-1)$-element axis.

Let us consider the product $\prod_{t\in T} P_t^*$ with product topology. The algebra of its open subsets K^*, generated by the product subbase, is a lattice of sets. It is obvious that it contains the subalgebra B^* of its Boolean (i.e., open and closed) elements isomorphic to A^*. But if E_j^t denotes the constant E_j in the algebra P_t^*, then, for $t_1 \neq t_2$ and $0 \neq j \neq n-1$, $p_{t_1}^{-1}(E_j^{t_1}) \neq p_{t_2}^{-1}(E_j^{t_2})$, which means that images of the same constants by natural embeddings of different lattices are different.

Let ψ be the canonical continuous mapping of the space $\prod_{t\in T} P_t^*$ onto the quotient space $\prod_{t\in T} P_t^* / \approx$. It is easy to see that, for $Y \in B^*$ and $Z \in K^*$, if $Z \neq Y$, then $\psi(Z) \neq \psi(Y)$ and $\psi(Y)$ is open. Hence the family $\{\psi(Y)\}_{Y\in B^*}$ is a lattice of open sets, isomorphic to A^*, i.e., a Boolean algebra.

The mapping ψ is open. Let us take a set from the subbase of $\prod_{t\in T} P_t^*$, that is a set of the form $\prod_{t\in T} Z_t$, where $Z_t = P_t^*$ for $t \neq t_0$ and Z_{t_0} belongs to the subbase of $P_{t_0}^*$. Thus Z_{t_0} has the form $X_{t_0} \times [1, n-1]$, where X_{t_0} belongs to the Stone subbase of $B_{t_0}^*$, or the form $B_{t_0}^* \times [1, j]$ for some $j < n-1$. In the first case,

$$\psi^{-1}\psi\Big(\prod_{t\in T} Z_t\Big) = \psi^{-1}\psi p_{t_0}^{-1}(X_{t_0} \times [1, n-1]) = p_{t_0}^{-1}(X_{t_0} \times [1, n-1]),$$

since $\prod_{t\in T} Z_t \in B^*$. In the second case,

$$\psi^{-1}\psi\Big(\prod_{t\in T} Z_t\Big) = \psi^{-1}\psi p_{t_0}^{-1}(B_{t_0}^* \times [1, j]) = \psi^{-1}\psi\big(\{a\colon f_{t_0} p_{t_0}(a) \leqslant j\}\big)$$

$$= \big\{a\colon \underset{t\in T}{\exists} f_t p_t(a) \leqslant j\big\} = \bigcup_{t\in T}\Big(\Big(\prod_{t'\in T\setminus\{t\}} B_{t'}^*\Big) \times (B_t^* \times [1, j])\Big)$$

and every set from this union is obviously open.

Moreover, in the first case,

$$i\Big(\prod_{t\in T} Z_t\Big) = g^{-1} p_{t_0}^{-1}(X_{t_0}),$$

and, in the second case,

$$i\Big(\prod_{t\in T} Z_t\Big) = f^{-1}([1, j]),$$

and sets of these two forms belong to the subbase of $A^* \times N^*$. Conversely, for every Z belonging to the subbase of $A^* \times N^*$, $i^{-1}(Z)$ is open. Thus i is a homeomorphism.

The second part of the lemma follows immediately from the first one. It is worth noticing that, for every t, t' and j, $p_t^{-1}(E_j^t) = p_{t'}^{-1}(E_j^{t'})$. The next theorem follows immediately from Lemma 2.

THEOREM 1. *If, for $t \in T$, P_t is a Post algebra of order n, then the algebra of open subsets of the reduced product $\prod_{t \in T} P_t^* / \approx$, generated by the sets of the form $\psi p_t^{-1}(Y)$, where Y is an element of the product subbase of $P_t^* = B_t^* \times N^*$, is the coproduct of algebras P_t.*

3. Free Post algebras. Now we are able to construct free Post algebras of order n (i.e., free algebras in the classes of Post algebras of order n for $n = 1, 2, \ldots$) [1] analogous to the Cantor discontinuum but somewhat more complicated.

LEMMA 3. *A Post free algebra of order n with one generator is a Post algebra $\boldsymbol{P}_n^*$ of all open subsets of an $n(n-1)$-point space $P_n^* = B_n^* \times N^*$ with the product topology, where $B_n^* = \{b_0, \ldots, b_{n-1}\}$ is the n-point discrete space. A free generator of $\boldsymbol{P}_n^*$ is the set*

$$G = \{b_1\} \times [1] \cup \{b_2\} \times [1, 2] \cup \ldots \cup \{b_{n-1}\} \times [1, n-1].$$

Proof. It follows from Lemma 1 that $\boldsymbol{P}_n^*$ is a Post algebra of order n. The set G generates the whole algebra $\boldsymbol{P}_n^*$, since each of its Boolean elements U can be written as a combination of operations on G:

$$U = \bigcup_{j=0}^{n-1} \big(\delta_j \cap \big(D_j(G) \setminus D_{j+1}(G)\big)\big),$$

where $D_0(G) = P_n^*$, $D_n(G) = \emptyset$ and $\delta_j = \emptyset$ or $\delta_j = P_n^*$ for every j. Every element of $\boldsymbol{P}_n^*$ can be obtained from Boolean elements and constants by means of Post operations.

Let φ be an embedding of G in a Post algebra $\boldsymbol{P}$ of order n. A Post homomorphism h of $\boldsymbol{P}_n^*$ into $\boldsymbol{P}$ we define as follows:

$$\begin{aligned}
h(\{b_{n-1}\} \times [1, n-1]) &= D_{n-1}\big(\varphi(G)\big), \\
h(\{b_{n-2}\} \times [1, n-1]) &= D_{n-2}\big(\varphi(G)\big) \setminus D_{n-1}\big(\varphi(G)\big), \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
h(\{b_1\} \times [1, n-1]) &= D_1\big(\varphi(G)\big) \setminus D_2\big(\varphi(G)\big), \\
h(\{b_0\} \times [1, n-1]) &= \neg D_1\big(\varphi(G)\big),
\end{aligned}$$

$$h(E_j) = e_j \quad \text{for } j = 0, \ldots, n-1.$$

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[1] For $n = 1$, we have the degenerated algebra and the empty set as its Stone space.

In such a way we have defined the mapping of every Boolean element and constant of P_n^* and thus of the whole algebra P_n^*. For G we have

$$h(G) = h\Big(\bigcup_{j=1}^{n-1} D_j(G) \cap E_j\Big) = \bigcup_{j=1}^{n-1} \big(h(D_j(G)) \cap h(E_j)\big)$$
$$= \bigcup_{j=1}^{n-1} \big(h(\{b_j, \ldots, b_{n-1}\} \times [1, n-1]) \cap e_j\big) = \bigcup_{j=1}^{n-1} D_j\big(\varphi(G)\big) \cap e_j = \varphi(G).$$

P_n^* is, therefore, a free Post algebra of order n with one generator. The number of its elements agrees with the Traczyk result (in a Post algebra P of order n there exist m independent elements if and only if P has at least n^m non-void and disjoint elements [10]). P_n^* is the simplest algebra which contains one independent element. Every independent element x in a Post algebra satisfies the condition: $i \neq j \Rightarrow D_i(x) \neq D_j(x)$ (cf. [10], Theorem 1). The condition is obviously satisfied by G in P_n^*.

In view of Lemma 3 and Theorem 1, we can prove the following

THEOREM 2. *For any cardinal $\varkappa$, the algebra of open sets of a reduced product $(P_n^*)^\varkappa / \approx$ of $\varkappa$ copies of the space P_n^*, generated as in Theorem* 1, *is a free Post algebra with $\varkappa$ generators of the form*

$$\psi p_i^{-1}\Big(\bigcup_{j=1}^{n-1} (\{b_j\} \times [1, j])\Big).$$

Proof. Let φ be any mapping of the set of free generators into some Post algebra P of order n. We define a homomorphism h on every subalgebra generated by one free generator as in Lemma 3. If we have defined h on elements Y and Z, we define it further as follows:

$$h(Y \cup Z) = h(Y) \cup h(Z), \quad h(Y \cap Z) = h(Y) \cap h(Z),$$
$$h\big(D_j(Y)\big) = D_j\big(h(Y)\big), \quad h(\neg Y) = \neg h(Y),$$
$$h(Y \rightarrow Z) = h(Y) \rightarrow h(Z).$$

It is worth noticing that every set of m free generators $\{a_j\}$ satisfies the Traczyk condition (see [10]) for independent elements

$$\bigcap_{j=1}^{m} C_{q(j)}(a_j) \neq \emptyset \qquad \text{for every } q \in \{0, \ldots, n-1\}^{\{1,\ldots,m\}},$$

where C_j are disjoint Boolean operators, i.e., $C_j(a) = D_j(a) \setminus D_{j+1}(a)$.

Moreover, in a finite reduced product of m copies of P_n^* there exist exactly m independent elements and, consequently, exactly n^m non-empty and disjoint (Boolean) elements.

Of course, we can represent a free Post algebra of order n with $\varkappa$ generators as an algebra of subsets of a product of $\varkappa$ copies of the n-point discrete space and one $(n-1)$-point set with the order topology and write it in the form of an algebra of some subsets of the set of functions as in [4], p. 91-92.

Let T be a set of power $\varkappa$ and $t_0 \notin T$. Let Φ be the set of all functions from $T \cup \{t_0\}$ into $N = \{0, \ldots, n-1\}$. Let $\boldsymbol{S}$ be the family of subsets of Φ including all sets of the form

$$S_{t,j} = \{f \in \Phi: f(t) = j\} \quad \text{for } t \in T \text{ and } j \in N,$$
$$S_{t_0,j} = \{f \in \Phi: f(t_0) \leqslant j\} \quad \text{for } j \in N \setminus \{0\}$$

and only these sets.

COROLLARY 3. *The algebra of open subsets of $\varkappa$, generated by $\boldsymbol{S}$, is a free Post algebra of order n with $\varkappa$ generators. The free generators have the form*

$$\bigcup_{j=1}^{n-1} (S_{t,j} \cap S_{t_0,j}) \quad \textit{for } t \in T.$$

The proof follows immediately from Theorems 1 and 2. In the case $n = 2$ this construction is equivalent to the construction of the Cantor discontinuum.

An analogical topological representation and construction of free algebras is applicable to some kinds of generalized Post algebras (e.g., [13]) and it will be the subject of another paper.

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