COLLOQUIUM MATHEMATICUM

VOL. LVII 1989 FASC. 1

ON WEAK UNIFORM DISTRIBUTION OF SEQUENCES OF INTEGERS

BY

E. ROSOCHOWICZ (POZNAŃ)

In [3] Zame has characterized those subsets $\mathcal{N}$ of the set of positive integers for which there exists a sequence of integers uniformly distributed modulo n if and only if $n \in \mathcal{N}$. The analogous problem can be posed for weak uniform distribution modulo n, $n \geq 3$ (see [2]), and we solve it here. I was informed that I. Z. Ruzsa also solved this problem using another approach ([1]).

I am obliged to Dr. E. Dobrowolski for his remarks. In particular, his suggestion of the choice of the function f_m occurring below essentially simplified my original proof.

We prove the following

THEOREM. *Let X be a set of integers ≥ 3 which has the following property*:

(*) *If $n \in X$ and m is a positive divisor of n with the same prime divisors (i.e. $\omega(m) = \omega(n)$), then $m \in X$.*

Then there exists a sequence $g_1, g_2, \ldots$ of integers which is weakly uniformly distributed modulo m (WUD(mod n)) *for all integers $n \in X$ and is not* WUD(mod k) *for all $k \notin X$* $(k \geq 3)$.

(Here $\omega(r)$ denotes the number of distinct prime factors of the integer r.)

Proof. We shall use the method employed by Zame [3] who considered the case of uniform distribution (mod n), and our fundamental tool will be the following lemma:

LEMMA. *Let $m \geq 1$ and let*

$$2 < n_1 < n_2 < \ldots < n_m, \quad 2 < k_1 < k_2 < \ldots < k_m$$

be integers with the property that for $1 \leq i, j \leq m$ we have either $\omega(k_i) \neq \omega(n_j)$ or $k_i \nmid n_j$ or both. Then:

([1]) I. Z. Ruzsa, *On sets of weak uniform distribution*, this fasc., pp. 183–187. [Note of the Editors]

(i) *There exists an infinite sequence* $x_1^{(m)}, x_2^{(m)}, \ldots$ *of integers which is* WUD(mod n_j) *but not* WUD(mod k_j) *for* $j = 1, 2, \ldots, m$.

(ii) *If, for* $i = 1, 2, \ldots, m$, ψ_i *is an arbitrary primitive character* (mod k_i) *if* $2\nmid k_i$, *and* (mod $(k_i/2)$) *if* $2 \,\|\, k_i$, *and we put*

$$\chi_i = \begin{cases} \psi_i & \text{if } 2\nmid k_i, \\ \chi_0^{(2)}\psi_i & \text{if } 2 \,\|\, k_i, \end{cases}$$

where $\chi_0^{(d)}$ *denotes the principal character* (mod d), *then*

$$\frac{1}{2^{i+2}} \leqslant \lim_{N\to\infty} \frac{\sum_{r=1}^{N} \chi_i(x_r^{(m)})}{\sum_{r=1}^{N} \chi_0^{(k_i)}(x_r^{(m)})} \leqslant \frac{1}{2^i}$$

holds for $i = 1, 2, \ldots, m$.

Proof of the Lemma. Let $t_m = \text{l.c.m.}(n_1, \ldots, n_m, k_1, \ldots, k_m)$, and for $k|t_m$ put

$$d_k = \prod_{\substack{p|t_m \\ p\nmid k}} p.$$

Let moreover A_d be the characteristic function of the set of all integers divisible by d. Note that, for $d|t_m$, A_d can be regarded as a function defined on $Z/t_m Z$. Now for every $m \geqslant 1$ we define a positive function f_m on $Z/t_m Z$ by putting

$$f_m = \frac{1}{2} + \sum_{i=1}^{m} \gamma_i A_{d_{k_i}} + \frac{1}{2}\sum_{i=1}^{m} \gamma_i A_{d_{k_i}} g_i,$$

where

$$g_i = \begin{cases} \chi_i + \bar{\chi}_i & \text{for } 2\nmid k_i \text{ and } k_i \neq 3, \\ \chi_i & \text{for } k_i = 3, \\ \chi_0^{(2)}\psi_i - A_2\psi_i & \text{for } k_i = 6, \\ (\chi_0^{(2)}\psi_i - A_2\psi_i) + (\chi_0^{(2)}\bar{\psi}_i - A_2\bar{\psi}_i) & \text{for } 2\|k_i \text{ and } k_i \neq 6 \end{cases}$$

and

$$\gamma_i = \frac{1}{2}\frac{d_{k_i}}{2^i}\left(\sum_{j=1}^{m} \frac{1}{2^j}\right)^{-1}.$$

Let μ_m be the normalized Haar measure on $Z/t_m Z$. Then, since $\chi_i \neq \chi_0^{(k_i)}$ and, for $2\|k_i$, $\psi_i \neq \chi_0^{(k_i/2)}$ $(i = 1, 2, \ldots, m)$, we have

$$\int_{Z/t_m Z} f_m \, d\mu_m = \frac{1}{2} + \sum_{i=1}^{m} \gamma_i d_{k_i}^{-1} = 1.$$

Let ν_m be the measure on $Z/t_m Z$ given by

$$d\nu_m = f_m \, d\mu_m.$$

Then ν_m is a probability measure, and therefore there exists a sequence $\{x_r^{(m)}\}$ in $Z/t_m Z$ which is uniformly distributed with respect to ν_m, i.e., such that for every complex-valued function g on $Z/t_m Z$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{r=1}^{N} g(x_r^{(m)}) = \int_{Z/t_m Z} g d\nu_m.$$

We shall now compute the value of the integral

$$\int_{Z/t_m Z} g f_m \, d\mu_m \quad \text{for } g = \chi_0^{(k_i)}, \chi_0^{(n_i)}, \chi_i, \chi^{(n_i)},$$

where $\chi^{(s)}$ denotes any character (mod s).

We need the following observation:

(i) (a) *If* χ_k *is a primitive character* (mod k), $l|t_m$, *and*

$$\int_{Z/t_m Z} A_{d_k} \chi_k \chi^{(l)} \neq 0,$$

then $k|l$, $\omega(k) = \omega(l)$ *and* $\bar{\chi}_k$ *induces* $\chi^{(l)}$.

(b) *If* $2||k_i$, $2|l$, *and*

$$\int_{Z/t_m Z} A_{d_{k_i}} \psi_i \chi^{(l)} d\mu_m \neq 0,$$

then $k_i|l$, $\omega(l) = \omega(k_i)$ *and* $\chi^{(l)}$ *is induced by* $\bar{\psi}_i$.

Proof. We start by noticing that

$$A_{d_k} = \prod_{p|d_k} (1 - \chi_0^{(p)}) = \sum_{d|d_k} (-1)^{\omega(d)} \chi_0^{(d)}. \tag{1}$$

Therefore

$$\int_{Z/t_m Z} A_{d_k} \chi_k \chi^{(l)} d\mu_m \neq 0 \Rightarrow \int_{Z/t_m Z} \chi_0^{(d)} \chi_k \chi^{(l)} d\mu_m \neq 0$$

for a certain $d|d_k$. This shows that for all x prime to $t = \text{l.c.m.}(d, k, l)$ we have $\bar{\chi}_k(x) = \chi^{(l)}(x)$; hence $\bar{\chi}_k$ and $\chi^{(l)}$ induce the same character (mod t). Since χ_k is primitive, $(k, l) = k$, i.e., $k|l$. Finally, the assumption implies that $A_{d_k}\chi^{(l)}$ does not vanish identically, so $(d_k, l) = 1$, and we infer that every prime divisor of l divides also k. Hence we must have $\omega(k) = \omega(l)$, and (a) is established.

In case (b) the proof follows the same line.

(ii) $\int_{Z/t_m Z} A_{d_{k_i}} \chi^{(l)} d\mu_m = 0$ *for* $\chi^{(l)} \neq \chi_0^{(l)}$.

This follows from (1).

(iii) *If* $2 \nmid k_i$ *and*

$$\int_{Z/t_m Z} A_{d_{k_i}} g_i \chi^{(l)} d\mu_m \neq 0,$$

then $k_i | l$, $\omega(l) = \omega(k_i)$, *and* $\chi^{(l)}$ *is induced by either* χ_i *or* $\bar{\chi}_i$.

This assertion is an immediate consequence of (i).

(iv) *If* $2 || k_i$ *and* $2 \nmid l$, *then*

$$\int_{Z/t_m Z} A_{d_{k_i}} g_i \chi^{(l)} d\mu_m = 0.$$

For the proof apply the definition of g_i and the following identity:

$$\int_{\{x \in Z/t_m Z: 2|x\}} A_{d_{k_i}} \psi_i \chi^{(l)} d\mu_m = \int_{\substack{\{y \in Z/t_m Z: \\ y = x + l d_{k_i} k_i/2, 2|x\}}} A_{d_{k_i}} \psi_i \chi^{(l)} d\mu_m = \int_{\{x \in Z/t_m Z: 2 \nmid x\}} A_{d_{k_i}} \psi_i \chi^{(l)} d\mu_m,$$

where $2 \nmid l d_{k_i}(k_i/2)$ and $l d_{k_i} k_i/2$ is the period of the function $A_{d_{k_i}} \psi_i \chi^{(l)}$.

From the properties of n_j and k_i $(1 \leqslant i, j \leqslant m)$ and (i)–(iv) we get:

(a) We have

$$\begin{aligned} &\int_{Z/t_m Z} f_m \chi_0^{(k_i)} d\mu_m \\ &= \frac{1}{2} \int_{Z/t_m Z} \chi_0^{(k_i)} d\mu_m + \sum_{j=1}^{m} \gamma_j \int_{Z/t_m Z} A_{d_{k_j}} \chi_0^{(k_i)} d\mu_m \\ &= \frac{1}{2} \frac{t_m}{k_i} \varphi(k_i) \frac{1}{t_m} + \sum_{j=1}^{m} \gamma_j \int_{\{x: d_{k_j} x \in Z/t_m Z\}} \chi_0^{(k_i)}(x) \chi_0^{(k_i)}(d_{k_j}) d\mu_m \\ &= \frac{1}{2} \frac{\varphi(k_i)}{k_i} + \sum_{j=1}^{m} \gamma_j \chi_0^{(k_i)}(d_{k_j}) \frac{\varphi(k_i)}{d_{k_j} k_i}, \end{aligned}$$

where if $\chi_0^{(k_i)}(d_{k_j}) \neq 0$, i.e., $(k_i, d_{k_j}) = 1$, then $k_i | (t_m/d_{k_j})$, and hence

$$\frac{1}{2} \frac{\varphi(k_i)}{k_i} \leqslant \int_{Z/t_m Z} f_m \chi_0^{(k_i)} d\mu_m \leqslant \frac{\varphi(k_i)}{k_i},$$

and similarly

$$\frac{1}{2} \frac{\varphi(n_j)}{n_j} \leqslant \int_{Z/t_m Z} f_m \chi_0^{(n_j)} d\mu_m \leqslant \frac{\varphi(n_j)}{n_j}.$$

(b) We have

$$\int_{Z/t_m Z} f_m \chi d\mu_m = 0$$

for any character $\chi \pmod{n_j}$.

(c) If $2\nmid k_i$, then since χ_i is a primitive character, we have

$$\int_{Z/t_mZ} f_m\chi_i\,d\mu_m = \frac{1}{2}\sum_{j=1}^{m}\gamma_j \int_{Z/t_mZ} A_{d_{k_j}} g_j\chi_i\,d\mu_m$$

$$\overset{\text{(i),(iii),(iv)}}{=} \frac{1}{2}\gamma_i \int_{Z/t_mZ} A_{d_{k_i}}\chi_0^{(k_i)}\,d\mu_m = \frac{1}{2}\gamma_i\frac{1}{d_{k_i}}\frac{\varphi(k_i)}{k_i}$$

$$= \frac{1}{2^{i+2}}\left(\sum_{j=1}^{m}\frac{1}{2^j}\right)^{-1}\frac{\varphi(k_i)}{k_i}.$$

(d) Similarly, for $2\,||\,k_i$ and $\chi_i = \chi_0^{(2)}\psi_i$, ψ_i being a primitive character $(\bmod\,(k_i/2))$, we have

$$\int_{Z/t_mZ} f_m\chi_i\,d\mu_m = \frac{1}{2}\sum_{j=1}^{m}\gamma_j \int_{Z/t_mZ} A_{d_{k_j}} g_j\chi_i\,d\mu_m$$

$$\overset{\text{(i),(iii)}}{=} \frac{1}{2}\gamma_i \int_{Z/t_mZ} A_{d_{k_i}}(1-2A_2)\chi_0^{(2)}\bar{\psi}_i\psi_i\,d\mu_m$$

$$= \frac{1}{2}\gamma_i \int_{Z/t_mZ} A_{d_{k_i}}\chi_0^{(k_i)}\,d\mu_m = \frac{1}{2^{i+2}}\left(\sum_{j=1}^{m}\frac{1}{2^j}\right)^{-1}\frac{\varphi(k_i)}{k_i}.$$

It follows that there exists a sequence $\{x_r^{(m)}\}$ in Z/t_mZ such that

(i*)
$$\lim_{N\to\infty}\frac{\sum_{r=1}^{N}\chi^{(n_j)}(x_r^{(m)})}{\sum_{r=1}^{N}\chi_0^{(n_j)}(x_r^{(m)})} = \frac{\int_{Z/t_mZ} f_m\chi^{(n_j)}\,d\mu_m}{\int_{Z/t_mZ} f_m\chi_0^{(n_j)}\,d\mu_m} = 0$$

for any $\chi^{(n_j)}$ $(\bmod\, n_j)$, $1\leqslant j\leqslant m$;

(ii*)
$$\lim_{N\to\infty}\frac{\sum_{r=1}^{N}\chi_j(x_r^{(m)})}{\sum_{r=1}^{N}\chi_0^{(k_j)}(x_r^{(m)})} = \frac{\int_{Z/t_mZ} f_m\chi_j\,d\mu_m}{\int_{Z/t_mZ} f_m\chi_0^{(k_j)}\,d\mu_m} = a_j^{(m)},$$

where

$$\frac{1}{2^{j+2}} \leqslant \frac{1}{2^{j+2}}\left(\sum_{i=1}^{m}\frac{1}{2^i}\right)^{-1} \leqslant a_j^{(m)} \leqslant \frac{1}{2^{j+1}}\left(\sum_{i=1}^{m}\frac{1}{2^i}\right)^{-1} \leqslant \frac{1}{2^j},$$
$$1\leqslant j\leqslant m.$$

The Lemma is thus proved.

Now we prove the Theorem. Let $X = \{n_1 < n_2 < \ldots\}$ be a set of positive integers $\geqslant 3$ satisfying $(*)$ and let $X' = \{k_1 < k_2 < \ldots\}$ be the sequence of all the remaining integers $\geqslant 3$. Observe that for any i, j we have either $\omega(k_i) \neq \omega(n_j)$ or $k_i \nmid n_j$ or both. Then by the Lemma for every $m \geqslant 1$ there exists a sequence $\{x_r^{(m)}\}$ of integers satisfying (i*) and (ii*), where if either X or X' is finite, we just take either $n_1, \ldots, n_{m'}$, $m' = \bar{X}$, or $k_1, \ldots, k_{m'}$, $m' = \bar{X}'$, suitable for each $m \geqslant m'$. (In fact, the Lemma is true for any m, $m' \geqslant 1$, $n_1, \ldots, n_{m'}$ and $k_1, \ldots, k_{m'}$.)

We enumerate the non-principal characters χ modulo the various n_j as $X_1, X_2, \ldots$, i.e.,

$$X_1 = \chi_1^{(n_1)}, \ldots, X_{\varphi(n_1)-1} = \chi_{\varphi(n_1)-1}^{(n_1)},$$

where $\chi_j^{(n_1)}$ are various, non-principal characters $(\bmod\, n_1)$,

$$X_{\varphi(n_1)} = \chi_1^{(n_2)}, \ldots, X_{\varphi(n_1)+\varphi(n_2)-2} = \chi_{\varphi(n_2)-1}^{(n_2)},$$

$\chi_j^{(n_2)}$ are various characters different from $\chi_0^{(n_2)}$, and so on. Then for $1 \leqslant j \leqslant m$ we obtain

$$\frac{\sum_{r=1}^{N} X_j(x_r^{(m)})}{\sum_{r=1}^{N} X_0^{(j)}(x_r^{(m)})} \to 0 \quad \text{as } N \to \infty,$$

where $X_0^{(j)}$ is the principal character of the same modulus as X_j. Thus we have

$$\forall_{i \geqslant m} \lim_{N\to\infty} \frac{\sum_{r=1}^{N} X_m(x_r^{(i)})}{\sum_{r=1}^{N} X_0^{(m)}(x_r^{(i)})} = 0,$$

$$\lim_{N\to\infty} \frac{\sum_{r=1}^{N} \chi_m(x_r^{(i)})}{\sum_{r=1}^{N} \chi_0^{(k_m)}(x_r^{(i)})} = a_m^{(i)},$$

$$\frac{1}{2^{m+2}} \leqslant |a_m^{(i)}| \leqslant \frac{1}{2^m},$$

$$\frac{1}{2}\frac{\varphi(k_m)}{k_m} \leqslant \lim_{N\to\infty} \frac{1}{N} \sum_{r=1}^{N} \chi_0^{(k_m)}(x_r^{(i)}) \leqslant \frac{\varphi(k_m)}{k_m},$$

and similarly

$$\frac{1}{2}R_m \leqslant \lim_{N\to\infty} \frac{1}{N} \sum_{r=1}^{N} X_0^{(m)}(x_r^{(i)}) \leqslant R_m,$$

i.e.,

$$\underset{m}{\forall}\ \underset{i\geqslant m}{\forall}\ \underset{s(i)}{\exists}\ \underset{N\geqslant s(i)}{\forall}\ \left|\frac{\sum_{r=1}^{N} X_m(x_r^{(i)})}{\sum_{r=1}^{N} X_0^{(m)}(x_r^{(i)})}\right| < \frac{1}{2^i},$$

$$\frac{1}{2^{m+3}} \leqslant \left|\frac{\sum_{r=1}^{N} \chi_m(x_r^{(i)})}{\sum_{r=1}^{N} \chi_0^{(k_m)}(x_r^{(i)})}\right| \leqslant \frac{1}{2^{m-1}},$$

$$\frac{1}{4}\frac{\varphi(k_m)}{k_m} \leqslant \frac{1}{N}\sum_{r=1}^{N} \chi_0^{(k_m)}(x_r^{(i)}) \leqslant 2\frac{\varphi(k_m)}{k_m},$$

$$\frac{1}{4}R_m \leqslant \frac{1}{N}\sum_{r=1}^{N} X_0^{(m)}(x_r^{(i)}) \leqslant 2R_m.$$

Let $r(i) \stackrel{\text{df}}{=} 2^i s(i+1)$ and let $2^{i+2}s(i) < s(i+1)$. Put

$$\{g_r\} = T_1 \cup T_2 \cup \ldots,$$

the elements of various finite sequences $T_1, T_2, \ldots$ written in the obvious order where

$$T_i = \{x_1^{(i)}, \ldots, x_{r(i)}^{(i)}\}$$

(i.e. $\{g_r\} = \{x_1^{(1)}, \ldots, x_{r(1)}^{(1)}, x_1^{(2)}, \ldots, x_{r(2)}^{(2)}, x_1^{(3)}, \ldots\}$).

We assert that $\{g_r\}$ is WUD(mod n) for each $n \in X$, i.e.,

$$\underset{m}{\forall} \lim_{N\to\infty} \frac{\sum_{r=1}^{N} X_m(g_r)}{\sum_{r=1}^{N} X_0^{(m)}(g_r)} = 0.$$

In fact,

$$\left|\frac{\sum_{r=1}^{N} X_m(g_r)}{\sum_{r=1}^{N} X_0^{(m)}(g_r)}\right| \leqslant \frac{\sum_{l=1}^{m-1}\left|\sum_{r=1}^{r(l)} X_m(x_r^{(l)})\right|}{\sum_{r=1}^{N} X_0^{(m)}(g_r)} + \frac{\sum_{l=m}^{m_N}\left|\sum_{r=1}^{r(l)} X_m(x_r^{(l)})\right|}{\sum_{r=1}^{N} X_0^{(m)}(g_r)} + \frac{\left|\sum_{r=1}^{M} X_m(x_r^{(m_N+1)})\right|}{\sum_{r=1}^{N} X_2^{(m)}(g_r)},$$

where

$$\sum_{i=1}^{m_N} r(i) \leqslant N < \sum_{i=1}^{m_N+1} r(i), \quad M = N - \sum_{i=1}^{m_N} r(i).$$

The first summand on the right-hand side of the above inequalitv converges to zero; the remaining two will be denoted by II and III, respectively.

For $l \geqslant m$ and large n we have

$$\Big|\sum_{r=1}^{n} X_m(x_r^{(l)})\Big| < \frac{\sum_{r=1}^{n} X_0^{(m)}(x_r^{(l)})}{2^l}.$$

Let

$$f(l) = \frac{\sum_{r=1}^{r(l)} X_0^{(m)}(x_r^{(l)})}{2^l}.$$

Then

$$\tfrac{1}{4}R_m s(l+1) \leqslant f(l) \leqslant 2R_m s(l+1) \quad \text{and} \quad f(l) \geqslant f(l-1).$$

Hence

$$\mathrm{II} \leqslant \frac{\sum_{l=m}^{m_N} \Big|\sum_{r=1}^{r(l)} X_m(x_r^{(l)})\Big|}{\sum_{l=1}^{m_N} \sum_{r=1}^{r(l)} X_0^{(m)}(x_r^{(l)})} < \frac{\sum_{l=m}^{m_N} f(l)}{\sum_{l=1}^{m_N} 2^l f(l)} \leqslant \frac{m_N f(m_N)}{2^{m_N} f(m_N)} \to 0 \quad \text{as } N \to \infty.$$

For $M \geqslant s(m_N+1)$,

$$\mathrm{III} = \frac{\Big|\sum_{r=1}^{M} X_m(x_r^{(m_N+1)})\Big|}{\sum_{r=1}^{N} X_0^{(m)}(g_r)} < \frac{1}{2^{m_N+1}} \to 0 \quad \text{as } N \to \infty.$$

For $M < s(m_N+1)$,

$$\mathrm{III} \leqslant \frac{\sum_{r=1}^{M} 1}{\sum_{r=1}^{N} X_0^{(m)}(g_r)} < \frac{s(m_N+1)}{2^{m_N} f(m_N)} \leqslant \frac{1}{2^{m_N-2} R_m} \to 0 \quad \text{as } N \to \infty.$$

To prove the Theorem it remains to show that for $k \notin X$ the sequence $\{g_r\}$ is not WUD(mod k). If

$$f(i) \stackrel{\mathrm{df}}{=} \sum_{r=1}^{r(i)} \chi_0^{(k_m)}(x_r^{(i)}) \cdot 2^{-i},$$

then

$$\frac{2^i f(i)}{2^{m+3}} \leqslant \Big|\sum_{r=1}^{r(i)} \chi_m(x_r^{(i)})\Big| \leqslant \frac{2^i f(i)}{2^{m-1}},$$

and furthermore

$$\frac{f(n-1)}{f(n)} \leqslant \frac{2s(n)}{\frac{1}{4}s(n+1)} \leqslant \frac{1}{2^{n-1}}.$$

For $N_n = \sum_{i=1}^{n} r(i)$ we have

$$\left|\frac{\sum_{r=1}^{N_n} \chi_m(g_r)}{\sum_{r=1}^{N_n} \chi_0^{(k_m)}(g_r)}\right| \geqslant \frac{\left|\sum_{r=1}^{r(n)} \chi_m(x_r^{(n)})\right| - \sum_{i=m}^{n-1}\left|\sum_{r=1}^{r(i)} \chi_m(x_r^{(i)})\right| - \sum_{i=1}^{m-1}\left|\sum_{r=1}^{r(i)} \chi_m(x_r^{(i)})\right|}{\sum_{i=1}^{n}\sum_{r=1}^{r(i)} \chi_0^{(k_m)}(x_r^{(i)})}$$

$$\geqslant \frac{1}{2^m}\frac{2^{n-3}f(n)-2\sum_{i=1}^{n-1} 2^i f(i)}{\sum_{i=1}^{n} 2^i f(i)} - o(n)$$

$$\geqslant \frac{1}{2^m}\frac{2^{n-3}f(n)-2^{2-n}f(n)\sum_{i=1}^{n-1} 2^i}{f(n)\sum_{i=1}^{n} 2^i} - o(n)$$

$$= \frac{1}{2^m}\frac{2^{n-3}-2^{2-n}\cdot 2(2^{n-1}-1)}{2(2^n-1)} - o(n)$$

$$\geqslant \frac{1}{2^m}\left(\frac{1}{2^4}-\frac{1}{2^{n-1}}\right) - o(n) \overset{n\to\infty}{\to} \frac{1}{2^{m+4}} > \frac{1}{2^{m+5}},$$

and so

$$\left|\frac{\sum_{r=1}^{N_n} \chi_m(g_r)}{\sum_{r=1}^{N_n} \chi_0^{(k_m)}(g_r)}\right| > \frac{1}{2^{m+5}}$$

for large n and $\{g_r\}$ is not WUD(mod k) for $k \notin X$.

REFERENCES

[1] P. C. Baayen and Z. Hedrlin, *On the existence of well-distributed sequences in compact spaces*, Indag. Math. 27 (1965), pp. 221–228.

[2] W. Narkiewicz, *On distribution of values of multiplicative function in residue classes*, Acta Arith. 12 (1967), pp. 269–279.

[3] A. Zame, *On a problem of Narkiewicz concerning uniform distributions of sequences of integers*, Colloq. Math. 24 (1972), pp. 271–273.

Reçu par la Rédaction le 28.6.1984