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OPERATOR SEMI-STABLE PROBABILITY MEASURES ON $\boldsymbol{R}^N$

BY

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Introduction. An *operator semi-stable measure* on $\boldsymbol{R}^N$ is a limit law for sums of the form $A_n(\xi_1+\ldots+\xi_{k_n})+h_n$, where ξ_i are $\boldsymbol{R}^N$-valued identically distributed random variables, A_n are non-singular linear operators in $\boldsymbol{R}^N$, $h_n \in \boldsymbol{R}^N$, and

$$\lim_{n\to\infty} k_{n+1}k_n^{-1} = r.$$

Such measures for $N = 1$ were considered by Kruglov [5] and in a multidimensional case by Jajte [4]. In order to define an operator semi-stable measure as a limit of a sequence of probability measures we need some preliminaries.

Let A be a linear operator in $\boldsymbol{R}^N$ and λ a Borel measure. We define a measure $A\lambda$ by the formula

$$A\lambda(E) = \lambda(A^{-1}E)$$

for all Borel subsets E of $\boldsymbol{R}^N$. For two probability measures μ and λ the following easily-verified equations hold:

$$(AB)\mu = A(B\mu), \quad A(\mu*\lambda) = A\mu*A\lambda, \quad (A\mu)^\wedge(x) = \hat{\mu}(A^*x),$$

where A and B are linear operators on $\boldsymbol{R}^N$, A^* denotes the adjoint operator, and $\hat{\mu}$ is the characteristic function of μ. For any Borel measure λ its support S_λ is a closed subset of $\boldsymbol{R}^N$ such that the complement of S_λ has λ-measure zero and $\lambda(U_x) > 0$ for any neighbourhood U_x of $x \in S_\lambda$. A measure λ is said to be *full* if its support is not contained in any $(N-1)$-dimensional hyperplane of $\boldsymbol{R}^N$.

Definition. A probability measure ν is said to be *operator semi-stable* if it is of the form

$$\nu = \lim_{n\to\infty} A_n\mu^{k_n}*\delta(h_n),$$

where A_n are non-singular ([1]) linear operators in $\boldsymbol{R}^N$, μ is a probability measure, $h_n \in \boldsymbol{R}^N$, and the sequence $k_1 < k_2 < \dots$ of positive integers is such that

$$\lim_{n\to\infty} k_{n+1}k_n^{-1} = r \quad \text{for some } r \ (1 \leqslant r < \infty).$$

Here "lim" denotes the weak limit of a sequence of probability measures and the power μ^{k_n} is taken in the sense of convolution.

1. A characteristic function of an operator semi-stable measure. Jajte proved in [4] the following theorem:

THEOREM 1.1. *A full probability measure μ on $\boldsymbol{R}^N$ is operator semi-stable if and only if it is infinitely divisible and there exist a number c $(0 < c < 1)$, a vector $h' \in \boldsymbol{R}^N$, and a non-singular linear operator B in $\boldsymbol{R}^N$ such that the formula*

$$\mu^c = B\mu * \delta(h') \tag{1.1}$$

holds. The spectrum of B is contained in the disc $\{|z|^2 \leqslant c\}$. The eigenvalues of B satisfying $|\lambda|^2 = c$ are simple. Furthermore, μ can be decomposed into the product $\mu = \mu_X * \mu_Y$ of two measures concentrated on B-invariant subspaces X and Y, respectively, such that $\boldsymbol{R}^N = X \oplus Y$, μ_X is a purely Poissonian operator semi-stable measure full on X, and μ_Y is a Gaussian measure full on Y. The spectrum of $B\,|\,X$ is contained in the disc $\{|z|^2 < c\}$ and the equality $|\lambda|^2 = c$ holds for the eigenvalues of $B\,|\,Y$.

Infinitely divisible measures for which the representation (1.1) holds for some c $(0 < c < 1)$ and a non-singular linear operator B such that $\mathrm{Sp}B \subset \{|z|^2 \leqslant c\}$ are said to be *quasi-decomposable* (by the pair (c, B)).

Remark 1.1. *If μ is quasi-decomposable, then it is quasi-decomposable by some pair (a, A) such that $\|A\| < 1$.*

Proof. Iterating (1.1) we obtain the equality $\mu^{c^n} = B^n\mu * \delta(h_n')$. By the inequality

$$\lim_{n\to\infty} \|B^n\|^{1/n} = r(B) \leqslant \sqrt{c} < 1,$$

where $r(B)$ denotes the spectral radius of B, we have $\|B^n\| \to 0$. Thus there exists an n_0 such that $\|B^{n_0}\| < 1$, and putting $A = B^{n_0}$ and $a = c^{n_0}$ we get $\mu^a = A\mu * \delta(h)$, $h \in \boldsymbol{R}^N$, $\mathrm{Sp}A \subset \{|z|^2 \leqslant a\}$, which means that μ is quasi-decomposable by the pair (a, A).

Remark 1.2. *If μ is quasi-decomposable, then it is operator semi-stable, even without the assumption that μ is full.*

([1]) As was pointed out by Professor C. Ryll-Nardzewski the word "non-singular" may be omitted and we get the same class of limit laws.

The proof is quite the same as the proof of the sufficiency of Theorem 1.1 in [4] (p. 35).

Now, let us recall the Lévy-Khintchine (L-K) representation of the characteristic function of an infinitely divisible measure μ:

$$\hat{\mu}(x) = \exp\left\{i(m, x) - \frac{1}{2}(Dx, x) + \int_{R^N\setminus\{0\}} K(x, y) M(dy)\right\}, \tag{1.2}$$

where $m \in R^N$, D is a non-negative linear operator in R^N, M an L-K spectral measure, i.e. a Borel measure on $R^N \setminus \{0\}$ finite outside every neighbourhood of zero and such that

$$\int_{0<\|x\|<1} \|x\|^2 M(dx) < \infty,$$

and K is defined by

$$K(x, y) = e^{i(x,y)} - 1 - \frac{i(x, y)}{1 + \|y\|^2}. \tag{1.3}$$

The representation (1.2) is unique and we write $\mu = [m, D, M]$ (see [7]).

LEMMA 1.1. *An infinitely divisible measure $\mu = [m, D, M]$ is quasi-decomposable by a pair (a, A) if and only if $aD = ADA^*$ and $aM = AM$.*

Proof. The equality $\mu^a = A\mu * \delta(h)$ is, in terms of characteristic functions, equivalent to the equality

$$\exp\left\{i(am, x) - \frac{1}{2}(aDx, x) + \int_{R^N\setminus\{0\}} K(x, y)(aM)(dy)\right\}$$

$$= \exp\left\{i(m', x) - \frac{1}{2}(ADA^*x, x) + \int_{R^N\setminus\{0\}} K(x, y)(AM)(dy)\right\}$$

which, according to the uniqueness of the L-K representation, is equivalent to $aD = ADA^*$ and $aM = AM$. Thus the proof is complete.

Let H be a real finite-dimensional Hilbert space, and A a non-singular linear operator acting in H such that $\|A\| < 1$. For an arbitrary $x \in H$, $x \neq 0$, we have

$$\|A^{n+1}x\| < \|A^n X\| \quad \text{for } n = 0, \pm 1, \ldots$$

Let us denote by Z_A the set of the form

$$Z_A = \{x \in H: \|x\| \leqslant 1\} \cap \{x \in H: \|A^{-1}x\| > 1\}. \tag{1.4}$$

If $\|x\| = 1$, then $x \in Z_A$ because of the inequality $\|A^{-1}x\| > 1$. Thus $Z_A \neq \emptyset$. For $x \in H, x \neq 0$, put

$$\tau_x = \{A^n x: n = 0, \pm 1, \ldots\}.$$

Of course, $\tau_x = \tau_y$ if and only if $y = A^k x$ for some integer k.

LEMMA 1.2. *If $x, y \in Z_A$ and $x \neq y$, then $\tau_x \neq \tau_y$ and $A^n x \notin Z_A$ for $n = \pm 1, \pm 2, \ldots$*

Proof. Let $x \in Z_A$. From the inequality $\|x\| = \|A^{-1}(Ax)\| \leqslant 1$ it follows that $Ax \notin Z_A$. For an arbitrary positive integer $n > 1$ we have

$$\|A^{-1}(A^n x)\| = \|A^{n-1}x\| < \|x\| \leqslant 1,$$

which means that $A^n x \notin Z_A$.

For $n < -1$ we have

$$\|A^n x\| = \|A^{-1}(A^{n+1}x)\| > \|A^{n+1}x\| \geqslant \|A^{-1}x\| > 1.$$

Of course, $A^{-1}x \notin Z_A$ because $\|A^{-1}x\| > 1$. Thus $A^n x \notin Z_A$ for an arbitrary integer $n \neq 0$.

Now, if $x, y \in Z_A$, then $A^n x = y$ only for $n = 0$, i.e. $x = y$, which completes the proof.

LEMMA 1.3. *For an arbitrary $x \neq 0$ there exists an integer n such that $A^n x \in Z_A$.*

Proof. Let us assume $x \notin Z_A$ and let, e.g., $\|x\| > 1$. The sequence $\{\|A^k x\|: k = 0, 1, \ldots\}$ tends monotonically to zero. Thus there exists an n such that $\|A^n x\| \leqslant 1$ and $\|A^{n-1}x\| > 1$, which means that $A^n x \in Z_A$. Now, for $\|A^{-1}x\| \leqslant 1$ we consider the sequence $\{\|A^{-k}x\|: k = 0, 1, \ldots\}$ tending monotonically to infinity and we obtain again $A^n x \in Z_A$ for some n.

From Lemmas 1.2 and 1.3 we get the following

COROLLARY 1.1. *We have*

$$H \setminus \{0\} = \bigcup_{n=-\infty}^{\infty} A^n Z_A \quad \text{and} \quad A^n Z_A \cap A^m Z_A = \emptyset \quad \text{for } n \neq m.$$

THEOREM 1.2. *Let A be a non-singular linear operator in H such that $\|A\| < 1$ and $\operatorname{Sp} A \subset \{|z|^2 < a\}$ for some $0 < a < 1$. A measure M is an L-K spectral measure in $H \setminus \{0\}$ satisfying the equality $aM = AM$ if and only if M is of the form*

$$M(E) = \sum_{n=-\infty}^{\infty} a^n \nu(A^n E \cap Z_A), \quad E \in \boldsymbol{B}(H \setminus \{0\}), \tag{1.5}$$

where ν is a finite Borel measure on Z_A. Moreover, the supports of M and ν are connected by the equality

$$S_M = \overline{\bigcup_{n=-\infty}^{\infty} A^n S_\nu}. \tag{1.6}$$

Proof. Let M be an L-K spectral measure satisfying $aM = AM$. By iterating we obtain

$$a^n M = A^n M \quad \text{for } n = 0, \pm 1, \pm 2, \ldots \tag{1.7}$$

For any Borel set $E \subset H \setminus \{0\}$ we have

$$M(E) = M\Big(\bigcup_{n=-\infty}^{\infty} (E \cap A^{-n} Z_A)\Big)$$
$$= \sum_{n=-\infty}^{\infty} (A^n M)(A^n E \cap Z_A) = \sum_{n=-\infty}^{\infty} a^n \nu(A^n E \cap Z_A),$$

where $\nu = M | Z_A$.

Now, let ν be a finite Borel measure on Z_A. We define the measure M by (1.5). It is easily seen that M is finite outside neighbourhoods of zero and satisfies the equality $aM = AM$. Moreover,

$$\int_{0<\|x\|<1} \|x\|^2 M(dx) = \sum_{n=0}^{\infty} \int_{A^n Z_A} \|x\|^2 M(dx)$$
$$= \sum_{n=0}^{\infty} \int_{Z_A} a^{-n} \|A^n x\|^2 M(dx)$$
$$\leqslant \int_{Z_A} \|x\|^2 M(dx) \sum_{n=0}^{\infty} \|A^n\|^2 a^{-n} < \infty.$$

The last inequality follows from the fact that

$$\lim_{n\to\infty} \sqrt[n]{\|A^n\|^2 a^{-n}} = \frac{[r(A)]^2}{a} < 1,$$

where $r(A)$ is the spectral radius of A. Thus M is an L-K spectral measure.

Now, let us assume that M and ν are connected by (1.5). Let $x_0 \in \bigcup_{n=-\infty}^{\infty} A^n S_\nu$ and V be an arbitrary neighbourhood of x_0 open in $H \setminus \{0\}$. There exists an n_0 such that $A^{n_0} x \in S_\nu$. Thus $A^{n_0} V \cap Z_A$ is a neighbourhood of $A^{n_0} x_0$ open in Z_A, which means that $\nu(A^{n_0} V \cap Z_A) > 0$. Consequently, $M(V) > 0$, and so $x_0 \in S_M$. Since the set $\bigcup_{n=-\infty}^{\infty} A^n S_\nu$ is of M-full measure, we get the inclusion

$$\bigcup_{n=-\infty}^{\infty} A^n S_\nu \subset S_M \subset \overline{\bigcup_{n=-\infty}^{\infty} A^n S_\nu}$$

which proves (1.6).

Now we prove a theorem characterizing quasi-decomposable measures.

THEOREM 1.3. *φ is a characteristic function of a quasi-decomposable probability measure on $\boldsymbol{R}^N$ if and only if*

$$(1.8)\quad \varphi(x) = \exp\Big\{i(m, x) - \frac{1}{2}(Dx, x) + \sum_{n=-\infty}^{\infty} a^{-n} \int_{Z_A \cap X} K(x, A^n y)\nu(dy)\Big\},$$

where $m \in \boldsymbol{R}^N$ and the following conditions are fulfilled:

(i) *A is a non-singular linear operator on $\boldsymbol{R}^N$ such that $\|A\| < 1$;*

(ii) *X is an A-invariant subspace of $\boldsymbol{R}^N$ such that* $\mathrm{Sp} A \,|\, X \subset \{|z|^2 < a\}$ *for some a $(0 < a < 1)$;*

(iii) *D is non-negative on $\boldsymbol{R}^N$ and satisfies the equality $aD = ADA^*$;*

(iv) *ν is a finite Borel measure on $Z_A \cap X$, where*

$$Z_A = \{x \in \boldsymbol{R}^N\colon \|x\| < 1\} \cap \{x \in \boldsymbol{R}^N\colon \|A^{-1}x\| > 1\}.$$

Moreover, φ determines m, D, and ν uniquely.

Proof. Sufficiency. Assume φ is of the form (1.8) and conditions (i)-(iv) are fulfilled. Let us define the measure M on X by (1.5). According to Theorem 1.2, M is an L-K spectral measure on X and (1.8) can be rewritten in the form

$$\varphi(x) = \exp\left\{i(m, x) - \frac{1}{2}(Dx, x) + \int\limits_{X\setminus\{0\}} K(x, y)\, M(dy)\right\},$$

which means that φ is a characteristic function of some infinitely divisible measure μ having its L-K spectral measure concentrated on the subspace X. By the equalities $aD = ADA^*$ and $aM = AM$ from Lemma 1.1, μ is quasi-decomposable.

Necessity. Let us assume φ is a characteristic function of some measure μ which is quasi-decomposable by the pair (a, A), where, according to Remark 1.1, we can take $\|A\| < 1$. If $\mu = [m, D, M]$, then from Lemma 1.1 we infer that $aM = AM$ and $aD = ADA^*$ and from Lemma 4 in [4] it follows that M is concentrated on an A-invariant subspace X of $\boldsymbol{R}^N$ such that $\mathrm{Sp} A \,|\, X \subset \{|z|^2 < a\}$. By Theorem 1.2, M has the representation (1.5) and, consequently,

$$\begin{aligned}\varphi(x) &= \exp\left\{i(m, x) - \frac{1}{2}(Dx, x) + \int\limits_{X\setminus\{0\}} K(x, y)\, M(dy)\right\} \\ &= \exp\left\{i(m, x) - \frac{1}{2}(Dx, x) + \sum_{n=-\infty}^{\infty} a^{-n} \int\limits_{Z_A \cap X} K(x, A^n y)\,\nu(dy)\right\},\end{aligned}$$

which proves the necessity.

The uniqueness of the representation (1.8) follows from the uniqueness of the L-K representation. As a consequence we obtain the following characterization of full operator semi-stable measures:

THEOREM 1.4. *φ is a characteristic function of a full operator semi-stable measure on $\boldsymbol{R}^N$ if and only if φ is of the form* (1.8) *and condi-*

tions (i)-(iii) *of Theorem* 1.3 *and the following conditions are satisfied*:

(iv′) ν *is a finite Borel measure on* $Z_A \cap X$ *such that*

$$\dim \overline{\bigcup_{n=-\infty}^{\infty} A^n S_\nu} = \dim X;$$

(v) *there exists an A-invariant subspace* Y *of* $\boldsymbol{R}^N$ *such that* $\boldsymbol{R}^N = X \oplus Y$, $DY = Y$, *and* $D|Y$ *is non-singular on* Y.

Moreover, φ *determines* m, D, *and* ν *uniquely.*

Proof. Necessity. Assume $\mu = [m, D, M]$ is a full operator semi-stable measure on $\boldsymbol{R}^N$. From Theorem 1.1 it follows that μ is quasi-decomposable, thus $\hat{\mu}$ is of the form (1.8), where conditions (i)-(iii) are fulfilled and $\mu = \mu_X * \mu_Y$. The fullness of μ_Y on Y implies condition (v) and the fullness of μ_X on X implies, according to (1.6), condition (iv′).

Sufficiency. Now assume φ is of the form (1.8) and (i)-(iii), (iv′), (v) are fulfilled. From Theorem 1.3 it follows that φ is a characteristic function of some quasi-decomposable measure $\mu = [m, D, M]$ which, according to Remark 1.2, is operator semi-stable. Condition (v) means that the measure μ_Y defined by

$$\hat{\mu}_Y(x) = \exp\left\{i(m_Y, x) - \frac{1}{2}(Dx, x)\right\}, \quad m = m_X + m_Y,\ m_X \in X,\ m_Y \in Y,$$

is full on Y. Condition (iv′) means that M is not concentrated on any proper subspace of X. Defining the measure μ_X by

$$\hat{\mu}_X(x) = \exp\left\{i(m_X, x) + \int_{X\setminus\{0\}} K(x, y)\, M(dy)\right\}, \quad x \in X,$$

we can easily verify that μ_X is full on X and $\mu = \mu_X * \mu_Y$ is full on $X \oplus Y = \boldsymbol{R}^N$. (Here we regard μ_X and μ_Y as defined on $\boldsymbol{R}^N$.) The uniqueness follows from the uniqueness of the L-K representation.

In the one-dimensional case we get the following representation of a characteristic function of a semi-stable measure.

COROLLARY 1.2. φ *is a characteristic function of a semi-stable measure on* $\boldsymbol{R}$ *if and only if either*

$$\varphi(t) = \exp\left\{itm - \frac{1}{2}\sigma^2 t^2\right\}$$

or

$$\varphi(t) = \exp\left\{itm + \sum_{n=-\infty}^{\infty} a^{-n} \int_{b<|x|\leqslant 1} \left(\exp\{itb^n x\} - 1 - \frac{itb^n x}{1 + b^{2n}x^2}\right) \nu(dx)\right\},$$

where $0 < a < 1$, $0 < b < a$, *and* ν *is a finite Borel measure on the set* $\{x : b < |x| \leqslant 1\}$.

2. Density of a full operator semi-stable measure on $\boldsymbol{R}^N$. Let μ be a full operator semi-stable measure on $\boldsymbol{R}^N$ and let, by Theorem 1.1, $\mu = \mu_X * \mu_Y$ be a decomposition of μ into full Poissonian and Gaussian measures. After simple computations we obtain

$$\hat{\mu}(z) = \hat{\mu}_X(z_X)\hat{\mu}_Y(z_Y),$$

where z_X and z_Y are the orthogonal projections of z onto X and Y, respectively. Since every element $z \in \boldsymbol{R}^N$ is uniquely determined by its projections, we can regard $\boldsymbol{R}^N$ as the Cartesian product $\boldsymbol{R}^N = X \times Y$ and write $z = (z_X, z_Y)$. Now let ν_X and ν_Y be Lebesgue measures on X and Y, respectively. Let $\tilde{\nu} = \nu_X \times \nu_Y$ be the product of the two measures. $\tilde{\nu}$ is a Borel measure on $\boldsymbol{R}^N$ and for every $z = (z_X, z_Y)$ and a set $E = E_X \times E_Y$, where E_X and E_Y are Borel subsets of X and Y, respectively, we have

$$\begin{aligned}\tilde{\nu}\big(E_X \times E_Y - (z_X, z_Y)\big) &= \nu_X(E_X - z_X)\nu_Y(E_Y - z_Y)\\ &= \nu_X(E_X)\nu_Y(E_Y) = \tilde{\nu}(E_X \times E_Y),\end{aligned}$$

which means that $\tilde{\nu}$ is invariant with respect to the group operation in $\boldsymbol{R}^N$. Thus $\tilde{\nu}$ is a Haar measure in $\boldsymbol{R}^N$ (see [2]) and, consequently, is of the form $\tilde{\nu} = k\nu$, where ν is the Lebesgue measure in $\boldsymbol{R}^N$. Hence

$$\int_{\boldsymbol{R}^N} |\hat{\mu}(z)|\nu(dz) = k^{-1} \int_X |\hat{\mu}_X(z_X)|\nu_X(dz_X) \int_Y |\hat{\mu}_Y(z_Y)|\nu_Y(dz_Y),$$

and since $\hat{\mu}_Y$ is the characteristic function of a full Gaussian measure on Y, $\hat{\mu}$ is Lebesgue-integrable on $\boldsymbol{R}^N$ if and only if $\hat{\mu}_X$ is Lebesgue-integrable on X.

THEOREM 2.1. *Let μ be a purely Poissonian operator semi-stable measure full on a finite-dimensional Euclidean space X. Then $\hat{\mu}$ is Lebesgue-integrable on X.*

We begin with the basic lemma:

LEMMA 2.1. *If μ is a purely Poissonian operator semi-stable measure full on X, then $|\hat{\mu}(x)| \neq 1$ for $x \neq 0$.*

Proof. Let us write

$$|\hat{\mu}(x)| = \left|\exp\Big\{\int_{X\setminus\{0\}} K(x, y)\, M(dy)\Big\}\right| = \exp\{-u(x)\},$$

where the kernel K is defined by (1.3) and

$$u(x) = \int_{X\setminus\{0\}} [1-\cos(x, y)]\, M(dy).$$

The L-K spectral measure M satisfies (1.7) with $\|A\| < 1$. We have $u(x) = 0$ if and only if $|\hat{\mu}(x)| = 1$, and let us assume $u(x_0) = 0$ for some

$x_0 \neq 0$. Hence we get $\cos(x_0, y) = 1$ for M-almost all y. Let us put $L_k = \{y: (x_0, y) = 2k\pi\}$ for $k = 0, \pm 1, \ldots$ Then the support S_M of M satisfies the inclusion

$$S_M \subset \bigcup_{k=-\infty}^{\infty} L_k.$$

By (1.7), S_M is A-invariant. Consequently, if $x \in S_M$, then

$$A^n x \in \bigcup_{k=-\infty}^{\infty} L_k$$

for any integer n. Since $A^n x \to 0$ as $n \to \infty$, we get $|(A^n x, x_0)| < \varepsilon$ for $n > n_0(\varepsilon)$, and taking ε sufficiently small we obtain $A^n x \in L_0$ for $n > n_0(\varepsilon)$. Now, if $\dim X = r$, then there exist $x_1, \ldots, x_r \in S_M$ and $x_1, \ldots, x_r$ are linearly independent (otherwise, M, and so μ, would not be full on X). There exists an m such that $A^m x_i \in L_0$ for $i = 1, \ldots, r$. The vectors $A^m x_1, \ldots, A^m x_r$ are linearly independent because A^m is non-singular, which means that $\dim L_0 = r$. Since L_0 is a subspace of X, $L_0 = X$, which is impossible because $x_0 \notin L_0$. Thus $u(x) = 0$ only for $x = 0$ and our lemma is proved.

Proof of Theorem 2.1. From the properties of M we obtain the following equality for the function u:

$$u(A^* x) = au(x). \tag{2.1}$$

Iterating (2.1) we get

$$u(A^{*n} x) = a^n u(x) \quad \text{for } n = 0, \pm 1, \ldots$$

Let λ be an arbitrary eigenvalue of A. Then there exists an m such that

$$a^m < |\lambda| < a^{1/2}, \quad a^{-1/2} < |\lambda|^{-1} < a^{-m}, \quad \lambda \in \operatorname{Sp} A, \tag{2.2}$$

and since $\operatorname{Sp} A^* = \{\bar{\lambda}: \lambda \in \operatorname{Sp} A\}$, inequalities (2.2) are valid for the eigenvalues of A^*. By the equality $\operatorname{Sp}(A^*)^{-1} = \{\lambda^{-1}: \lambda \in \operatorname{Sp} A^*\}$ we get $r\big((A^*)^{-1}\big) < a^{-m}$, where $r\big((A^*)^{-1}\big)$ is the spectral radius of $A^{*\,-1}$. Let Z_{A^*} be the set defined by (1.4) but for A^* instead of A and for X instead of H. The function u is continuous and $u(x) \neq 0$ for $x \in \bar{Z}_{A^*}$. Thus

$$\inf_{x \in Z_{A^*}} u(x) > 0$$

and, consequently, there exists a $k > 0$ such that $u(x) \geqslant k \|x\|^{1/m}$ for $x \in Z_{A^*}$. Since

$$\lim_{n \to \infty} \big\|\big((A^{-1})^*\big)^n\big\|^{1/n} = r\big((A^*)^{-1}\big) < a^{-m},$$

we obtain $\|(A^*)^{-n}\| < a^{-mn}$ for $n \geqslant n_0$. Thus for $n \geqslant n_0$ and $x \in Z_{A^*}$ we have

$$u\big((A^*)^{-n}x\big) = a^{-n}u(x) \geqslant a^{-n}k\|x\|^{1/m} = k(a^{-mn}\|x\|)^{1/m} \geqslant k\|(A^*)^{-n}x\|^{1/m}.$$

The last inequality means that

$$u(x) \geqslant k\|x\|^{1/m} \quad \text{for } x \in \bigcup_{n=n_0}^{\infty} (A^*)^{-n}Z_{A^*}. \tag{2.3}$$

Certainly, the set $\bigcup_{n=-\infty}^{n_0} (A^*)^{-n}Z_{A^*}$ is bounded, so it is contained in some ball $\{x\colon \|x\| \leqslant S\}$ and, for the x's lying outside the ball inequality (2.3) holds. Thus we obtain the estimation

$$|\hat{\mu}(x)| \leqslant \begin{cases} 1 & \text{for } \|x\| \leqslant S, \\ \exp\{-k\|x\|^{1/m}\} & \text{for } \|x\| > S \end{cases} \tag{2.4}$$

from which we infer that $\hat{\mu}$ is Lebesgue-integrable. Thus the proof is complete.

It is well known (see, e.g., [1]) that the integrability of $\hat{\mu}$ implies the existence of the density of μ. Thus we have

THEOREM 2.2. *A full operator semi-stable measure on $\boldsymbol{R}^N$ has a density.*

Estimation (2.4) implies the following remark due to Professor C. Ryll-Nardzewski:

Remark 2.1. *The density of a full operator semi-stable measure on $\boldsymbol{R}^N$ is of the class C^∞ and all its derivatives are bounded.*

In [8] Sharpe considered full operator stable measures. From the fact that every full operator stable measure is semi-stable we infer the following

COROLLARY 2.1. *A full operator stable measure on $\boldsymbol{R}^N$ has a density which is of the class C^∞ and all its derivatives are bounded.*

3. Absolute moments of an operator semi-stable measure. Let $\mu = [m, D, M]$ be an infinitely divisible measure on $\boldsymbol{R}^N$ and e a non-zero vector in $\boldsymbol{R}^N$. Define a random variable ξ on $\boldsymbol{R}^N$ by $\xi(x) = (x, e)$.

LEMMA 3.1. *The induced measure μ_ξ on the line is infinitely divisible with the L-K spectral measure M_ξ, where $M_\xi(E) = M(\xi^{-1}E)$.*

The lemma is a consequence of the equality $\hat{\mu}_\xi(t) = \hat{\mu}(te)$.

The following lemma is a generalization of the one-dimensional case from [6].

LEMMA 3.2. *Let $\mu = [m, D, M]$ be an infinitely divisible measure on $\boldsymbol{R}^N$. Then*

$$\int_{\boldsymbol{R}^N} \|x\|^a \mu(dx) < \infty$$

if and only if

$$\int_{\|x\|>1} \|x\|^a M(dx) < \infty .$$

Proof. Let $\{e_1, \ldots, e_N\}$ be an orthonormal basis in $\boldsymbol{R}^N$. For the sake of simplicity we put

$$\|x\| = \sum_{i=1}^{N} |(x, e_i)| .$$

By the inequalities

$$\begin{gathered} a_1^a + \ldots + a_N^a \leqslant (a_1 + \ldots + a_N)^a \leqslant N^{a-1}(a_1^a + \ldots + a_N^a), \qquad a \geqslant 1, \\ N^{a-1}(a_1^a + \ldots + a_N^a) \leqslant (a_1 + \ldots + a_N)^a \leqslant a_1^a + \ldots + a_N^a, \qquad 0 < a < 1, \\ \text{for } a_1, \ldots, a_N \geqslant 0, \end{gathered} \tag{3.1}$$

the integral $\int_{\boldsymbol{R}^N} \|x\|^a \mu(dx)$ is finite if and only if for every $i = 1, \ldots, N$ the integral $\int_{\boldsymbol{R}^N} |(x, e_i)|^a \mu(dx)$ is finite, which is equivalent to the condition

$$\int_{\boldsymbol{R}} |u|^a \mu_{\xi_i}(du) < \infty .$$

By Lemma 3.1 and [6] the last inequality holds if and only if for every $i = 1, \ldots, N$ the integral $\int_{|u|>1} |u|^a M_{\xi_i}(du)$ is finite, that is, if all the integrals

$$\int_{\{x: |(x, e_i)|>1\}} |(x, e_i)|^a \, M(dx) \qquad \text{for } i = 1, \ldots, N$$

are finite. Because of the inclusions

$$\{x\colon \|x\| > N\} \subset \bigcup_{i=1}^{N} \{x\colon |(x, e_i)| > 1\} \subset \{x\colon \|x\| > 1\}$$

and the finiteness of M outside neighbourhoods of zero this fact is equivalent to the finiteness of the integral $\int_{\|x\|>1} \|x\|^a \, M(dx)$, which completes the proof.

Let μ be a full operator semi-stable measure on $\boldsymbol{R}^N$. According to Theorem 1.1, μ is quasi-decomposable by some pair (c, B). Let us define a number δ by $c = |\lambda_{\min}|^\delta$, where $\lambda_{\min}$ is the eigenvalue of B having the smallest absolute value. Then we have

THEOREM 3.1. *The measure μ has absolute moments of order a for $a < \delta$ and has no absolute moments of order a for $a \geqslant \delta$.*

Proof. Let us introduce, as in Remark 1.1, the number a, the operator A, and define the set Z_A by (1.4). We have

$$\int_{\|x\|>1} \|x\|^a M(dx) = \sum_{n=1}^{\infty} \int_{A^{-n}Z_A} \|x\|^a M(dx) = \sum_{n=1}^{\infty} a^n \int_{Z_A} \|A^{-n}x\|^a M(dx).$$

Now we consider A^{-1} as an operator acting in the N-dimensional complex Euclidean space C^N being a natural extension of R^N. By the Jordan theorem on the canonical representation (see [3]) there exist a basis $\{f_1, \ldots, f_N\}$ in C^N, a system of integers $0 = n_0 < n_1 < \ldots < n_k = N$, and a sequence of eigenvalues $\theta_1, \ldots, \theta_k$ of A^{-1} indexed so that

$$A^{-1}f_i = \theta_j f_i + f_{i+1} \quad \text{for } n_{j-1} < i < n_j,$$
$$A^{-1}f_{n_j} = \theta_j f_{n_j} \quad \text{for } j = 1, \ldots, k.$$

Let us establish in R^N the norm putting

$$|||x||| = \sum_{l=1}^{N} |\beta_l(x)| \quad \text{for } x = \sum_{l=1}^{N} \beta_l(x) f_l.$$

After some computations we obtain

$$|||A^{-n}x||| = \sum_{j=1}^{k} \sum_{l=1}^{n_j - n_{j-1}} \left| \sum_{r=0}^{l} \binom{n}{r} \theta_j^{n-r} \right| |\beta_{n_{j-1}+l}(x)|.$$

Of course,

$$\int_{\|x\|>1} \|x\|^a M(dx) < \infty$$

if and only if

$$\int_{\|x\|>1} |||x|||^a M(dx) < \infty,$$

which, by (3.1), is equivalent to the condition

$$\begin{aligned} (3.2) \qquad & \sum_{n=1}^{\infty} a^n \sum_{j=1}^{k} \sum_{l=1}^{n_j-n_{j-1}} \left| \sum_{r=0}^{l} \binom{n}{r} \theta_j^{n-r} \right|^a \int_{Z_A} |\beta_{n_{j-1}+l}(x)|^a M(dx) \\ & = \sum_{j=1}^{k} \sum_{l=1}^{n_j-n_{j-1}} \sum_{n=1}^{\infty} a^n |\theta_j|^{na} \left| \sum_{r=0}^{l} \binom{n}{r} \theta_j^{-r} \right| \int_{Z_A} |\beta_{n_{j-1}+l}(x)|^a M(dx) < \infty. \end{aligned}$$

Now, let θ_{j_0} be the eigenvalue of A^{-1} having the greatest absolute value. If

$$\int_{Z_A} |\beta_{n_{j_0-1}+l}(x)|^a M(dx) = 0 \quad \text{for every } l = 1, \ldots, n_{j_0} - n_{j_0-1},$$

then

$$M(\{x \in Z_A:\ \beta_{n_{j_0-1}+l}(x) = 0,\ l = 1, \ldots, n_{j_0} - n_{j_0-1}\}) = M(Z_A),$$

and by virtue of the A^{-1}-invariancy of the subspace

$$X' = \{x \in \boldsymbol{R}^N:\ \beta_{n_{j_0-1}+l}(x) = 0,\ l = 1, \ldots, n_{j_0} - n_{j_0-1}\}$$

the measure M would be concentrated on X'. Since μ_X, and so M, is full on X by Theorem 1.1, we have $X \subset X'$.

The relation $\theta_{j_0} \in \mathrm{Sp}\, A^{-1}|X$ implies $f_{n_{j_0}} \in \tilde{X} \subset \tilde{X}'$, where $\tilde{X}$ and $\tilde{X}'$ denote complex extensions of X and X', respectively. But $f_{n_{j_0}} \notin \tilde{X}'$ by the definition of X', from which it follows that for at least one l ($l = 1, \ldots, n_{j_0} - n_{j_0-1}$)

$$\int_{Z_A} |\beta_{n_{j_0-1}+l}(x)|^a M(dx) > 0.$$

Thus the left-hand side of (3.2) is finite if and only if $a|\theta_{j_0}|^a < 1$. In other words, $a/|\lambda_{j_0}|^a < 1$, where λ_{j_0} is the eigenvalue of A having the smallest absolute value. The last inequality is equivalent to $c/|\lambda_{\min}|^a < 1$, which proves our theorem.

Applying this result to full operator stable measures having the representation

$$\mu^t = t^B \mu * \delta(h_t) \quad \text{for } t > 0 \tag{3.3}$$

(see [8]) we obtain

COROLLARY 3.1. *A full operator stable measure μ on $\boldsymbol{R}^N$ has absolute moments of order a for $a < \delta$ and has no absolute moments of order a for $a \geqslant \delta$, where $\delta = 1/\mathrm{Re}\,\lambda_{\max}$ and $\lambda_{\max}$ is the eigenvalue of B from* (3.3) *having the greatest real part.*

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