COLLOQUIUM MATHEMATICUM

XLII DÉDIÉ À LA MÉMOIRE D'EDWARD MARCZEWSKI 1979

A PLANCHEREL MEASURE FOR THE DISCRETE HEISENBERG GROUP

BY

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Let G be a locally compact group and let $\hat{G}$ denote the dual object of G, that is, the set of equivalence classes of irreducible unitary representations, equipped with the usual topology. Assume that G is unimodular and separable. If, moreover, G is a type I group (or equivalently: the topology of $\hat{G}$ is T_0), then the Plancherel theorem, due to Mautner and Segal ([1], 18.8.2), says that there exists a unique positive (σ-additive) measure μ on $\hat{G}$ such that, for all $f \in L^1(G) \cap L^2(G)$,

$$\int_G |f(x)|^2 dx = \int_{\hat{G}} \operatorname{Tr}\left(\pi(f)^* \pi(f)\right) d\mu(\pi), \tag{1}$$

Tr being the trace of operators in the Hilbert space $\mathscr{H}_\pi$ corresponding to π.

In practice, for a specific G it can be a very delicate problem to find μ explicitly (in terms of the group structure) and for many groups it is not yet done.

For groups which are not of type I, the situation is much more difficult and less investigated. An analogue of the Plancherel theorem need not be true. Such a measure need not exist (in my opinion, the discrete Heisenberg group is an example) or may not be unique. The difficulties appear already in the description of the space $\hat{G}$.

In this paper we give a simple construction of a finitely additive Plancherel measure μ on the dual object $\hat{G}$ of the discrete Heisenberg group such that all representations in the support of μ are of finite dimension and so have very simple form. This may allow us to use formula (1) in practice.

The group G of all matrices

$$\begin{pmatrix} 1 & n_1 & n_3 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix},$$

where n_1, n_2 and n_3 are integers, is called the *discrete Heisenberg group*. It is isomorphic to the set Z^3, Z being the integers, with the multiplication law defined by the formula

$$nm = (n_1+m_1, n_2+m_2, n_3+m_3+n_1m_2), \qquad n, m \in Z^3.$$

THEOREM. *Let G denote the discrete Heisenberg group. There exists a positive, finitely additive measure μ on $\hat{G}$, supported by the set of finite-dimensional representations of G, such that for every $f \in l^1(G)$ the function*

$$\pi \to \frac{1}{\dim \pi} \operatorname{Tr}\big(\pi(f)^* \pi(f)\big)$$

is μ-integrable on $\hat{G}$ and

$$\sum_{n \in G} |f(n)|^2 = \int_{\hat{G}} \frac{1}{\dim \pi} \operatorname{Tr}\big(\pi(f)^* \pi(f)\big) d\mu(\pi).$$

Let Q denote the set of all rational numbers in the interval $[0, 1)$, and let X be the space $[0, 1) \times [0, 1) \times Q$ equipped with the natural topology from R^3. Let $\mathcal{M}$ denote the field of subsets in X, generated by all "rectangles" $[a_1, b_1) \times [a_2, b_2) \times \big([a_3, b_3) \cap Q\big)$, where a_i, b_i $(i = 1, 2, 3)$ are arbitrary rational numbers satisfying $0 \leqslant a_i < b_i \leqslant 1$. It is easy to see that every element in the field $\mathcal{M}$ has a unique representation as a finite sum of pairwise disjoint rectangles. Thus there is a positive, finitely additive measure m on $\mathcal{M}$ which takes the value $(b_1-a_1)(b_2-a_2)(b_3-a_3)$ on the rectangle $[a_1, b_1) \times [a_2, b_2) \times \big([a_3, b_3) \cap Q\big)$.

LEMMA 1. *Let f be a uniformly continuous, complex-valued function on X. Define a function F_f on X by*

$$F_f(x_1, x_2, x_3) = \frac{1}{q^2} \sum_{k=0}^{q-1} \sum_{m=0}^{q-1} f\left(\frac{x_1+k}{q}, \frac{x_2+m}{q}, x_3\right)$$

if $x_3 = p/q$ with p and q relatively prime natural numbers. Then the function F_f is m-integrable on X and

$$\int_X F_f dm = \int_{[0,1]^3} \tilde{f},$$

where $\tilde{f}$ is the unique extension of f to a continuous function on $[0, 1]^3$ and the right-hand side of the last equality is the usual Riemann integral of $\tilde{f}$ on $[0, 1]^3$.

The proof of the lemma can be easily obtained from the following three observations.

(I) *A complex-valued bounded function g on X is m-integrable on X if and only if it can be extended to a Riemann integrable function $\tilde{g}$ on $[0, 1]^3$. In this case*

$$\int_X g\,dm = \int_{[0,1]^3} \tilde{g}.$$

Indeed, suppose that the function g is m-integrable. We may assume that g is real-valued. In any point $x_0 \in [0, 1]^3 - X$ let us define $\tilde{g}(x_0) = a$ with

$$\varliminf_{x \to x_0} g(x) \leqslant a \leqslant \varlimsup_{x \to x_0} g(x).$$

Then for every partition $\mathscr{P}$ of $[0, 1]^3$ the upper and lower integral sums of $\tilde{g}$ coincide with the corresponding integral sums of g on X with respect to the measure m. This implies that the function $\tilde{g}$ is integrable and

$$\int_{[0,1]^3} \tilde{g} = \int_X g\,dm.$$

The converse implication is trivial.

(II) *A bounded function $F\colon [0, 1]^3 \to C$ is Riemann-integrable if and only if the set of those points where F is not continuous has Lebesgue measure zero* (see [2], Theorem 3.8).

(III) *Let us extend F_f to a function $\tilde{F}_f$ on $[0, 1)^3$ by putting*

$$\tilde{F}_f(x_1, x_2, x_3) = \int_0^1 \int_0^1 \tilde{f}(t_1, t_2, x_3)\,dt_1\,dt_2$$

if x_3 is irrational. Then $\tilde{F}_f$ is continuous in every point of the extension.

Indeed, the integral

$$\int_0^1 \int_0^1 \tilde{f}(x_1, x_2, x_3)\,dx_1\,dx_2$$

is a continuous function of x_3. On the other hand, by the definition of the Riemann integral, for any $\varepsilon > 0$ and any rational x_3 we have

$$\left| \int_0^1 \int_0^1 f(x_1, x_2, x_3)\,dx_1\,dx_2 - \frac{1}{q^2} \sum_{k=0}^{q-1} \sum_{m=0}^{q-1} f\left(\frac{x_1+k}{q}, \frac{x_2+m}{q}, x_3\right) \right| < \varepsilon$$

for large q. In particular, the inequality

$$\left| \tilde{F}_f(x_1, x_2, x_3) - \int_0^1 \int_0^1 \tilde{f}(x_1, x_2, x_3)\,dx_1\,dx_2 \right| \geqslant \varepsilon$$

holds only for finitely many x_3. It follows that $\tilde{F}_f$ is continuous in every point $(x_1, x_2, x_3) \in [0, 1)^3$ with x_3 irrational.

For every $x = (x_1, x_2, x_3)$ in X, with $x_3 = p/q$ (p and q relatively prime), let us define a continuous finite-dimensional representation π_x of the group G, acting on the Hilbert space $\mathscr{H}_x$ of dimension q ($\mathscr{H}_x$ will be realized as a space of all complex, periodic sequences $\{\varphi(k)\}_{k\in Z}$ with the period q) as follows:

$$\big(\pi_x(n)\varphi\big)(k) = \exp\frac{2\pi i}{q}\,(n_1x_1+n_2x_2+n_3qx_3+kn_2qx_3)\varphi(k+n_1).$$

The reader can easily verify the following

LEMMA 2. *Every representation π_x is irreducible and, for $x_1 \neq x_2$ in X, representations π_{x_1} and π_{x_2} are inequivalent.*

Remark. It can be shown that $\{\pi_x\}_{x\in X}$ is the class of all finite-dimensional, irreducible, unitary representations of G up to the unitary equivalence.

LEMMA 3. *Fix an x in X and describe $x_3 = p/q$ in lowest terms. Then or every $f\in l^1(G)$ we have*

$$\operatorname{Tr}\big(\pi_x(f)^*\pi_x(f)\big) = \frac{1}{q}\sum_{k=0}^{q-1}\sum_{m=0}^{q-1}\left|\hat{f}\left(\frac{x_1+k}{q}, \frac{x_2+m}{q}, x_3\right)\right|^2,$$

where $\hat{f}$ is the usual Fourier transform of f on Z^3,

$$\hat{f}(y) = \sum_{n\in Z^3} f(n)e^{2\pi i\langle n,y\rangle}, \qquad y\in[0,1)^3.$$

Proof. Consider the orthonormal basis of $\mathscr{H}_x$ consisting of the periodic sequences $\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_{q-1}$, where $\varphi_k(m) = 1$ if $m\equiv k \pmod q$ and $\varphi_k(m) = 0$ if $m \not\equiv k \pmod q$. An easy calculation shows that the matrix of the operator $\pi_x(f)$ with respect to the above basis has coefficients $(a_{k,m})_{k,m=0}^{q-1}$ of the form

$$a_{k,m} = \sum_{n\in Z^3} f(n_1q+k-m, n_2, n_3)\exp\frac{2\pi i}{q}\,(n_1qx_1+kx_1-mx_1+ +n_2x_2+n_2mqx_3+n_3qx_3).$$

Furthermore,

$$\operatorname{Tr}\big(\pi_x(f)^*\pi_x(f)\big) = \sum_{k=0}^{q-1}\sum_{m=0}^{q-1}|a_{k,m}|^2.$$

One can show that for any integer s there is

$$\sum_{k=0}^{q-1} a_{k,m}\exp 2\pi i\frac{ks}{q} = \hat{f}\left(\frac{x_1+s}{q}, \frac{x_2}{q}+kx_3, x_3\right)\exp 2\pi i\,\frac{mx_1}{q}$$

which, by the formula

$$\sum_{s=0}^{q-1}\left|\sum_{k=0}^{q-1} a_{k,m}\exp 2\pi i\,\frac{ks}{q}\right|^2 = q\sum_{k=0}^{q-1}|a_{k,m}|^2,$$

gives the desired equality

$$\mathrm{Tr}\big(\pi_x(f)^*\pi_x(f)\big) = \sum_{k=0}^{q-1}\sum_{m=0}^{q-1}|a_{k,m}|^2 = \frac{1}{q}\sum_{k=0}^{q-1}\sum_{m=0}^{q-1}\left|\hat{f}\left(\frac{x_1+k}{q},\frac{x_2+m}{q},x_3\right)\right|^2.$$

Proof of the Theorem. By Lemma 2 the map $X \ni x \to \pi_x \in \hat{G}$ is an injection of X into $\hat{G}$, and so it transposes the measure m into a positive measure (say) μ on $\hat{G}$. Suppose now that f is in $l^1(G)$. Then, by Lemmas 3 and 1.

$$\int_{\hat{G}}\frac{1}{\dim\pi}\,\mathrm{Tr}\big(\pi(f)^*\pi(f)\big)\,d\mu(\pi) = \int_X \frac{1}{\dim\pi_x}\,\mathrm{Tr}\big(\pi_x(f)^*\pi_x(f)\big)\,dm(x)$$
$$= \int_X F_{|\hat{f}|^2}(x)\,dm(x) = \int_{[0,1]^3}|\hat{f}|^2.$$

By the Parseval formula the last integral is equal to

$$\sum_{n\in Z^3}|f(n)|^2,$$

and the Theorem follows.

REFERENCES

[1] J. Dixmier, *Les C*-algèbres et leurs représentations*, Paris 1969.
[2] M. Spivak, *Calculus on manifolds*, Mathematics monograph series, 1965.

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Reçu par la Rédaction le 3. 6. 1978