## ON THE RELATION
## BETWEEN CONTINUOUS AND CONNECTED FUNCTIONS*

BY

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A real-valued function $f$ defined on a topological space $X$ is said to be a *connected function* if it maps every connected set in its domain onto a connected set. If $X$ is an interval of real numbers, a connected function is called a *Darboux function*. Darboux functions have been extensively studied and a lengthy bibliography appears in Bruckner and Ceder [3].

Since every continuous real-valued function defined on an interval is a Darboux function, the question naturally arises as to when Darboux functions are continuous. It was Darboux himself who demonstrated that Darboux functions are nót necessarily continuous [4]. Gillespie [6] proved that if $f$ is Darboux and $\text{int}\{y:f^{-1}(y) \text{ is infinite}\} = \emptyset$, then $f$ is continuous. However, the converse of this statement is not true. A corollary (proved by Tricomi [12] and Jacobsthal [8]) is that a one-to-one Darboux function is continuous and monotonic. A necessary and sufficient condition that a Darboux function $f$ is continuous is that $f^{-1}(y)$ is closed for each $y \in f(X)$ [2].

Darboux functions have been generalized by changing conditions on the domain $X$ or on the range $Y$ or by requiring that $f(B)$ be a connected subset of $Y$ for $B \in \mathscr{B}$ where $\mathscr{B}$ is a certain class of connected subsets of $X$ (see [2], [5], [7], [9], [10], and [11]). The condition that $f^{-1}(y)$ is closed for each $y \in f(X)$ is no longer a sufficient condition that a generalized Darboux function is continuous [2].

Lipiński [9], in attempting to give necessary and sufficient conditions for a connected function to be continuous made the following definition. A real-valued function $f$ defined on a topological space $X$ is said to satisfy *property* $(G)$ if $f^{-1}(s)$ is closed for each $s \in S$, where $S$ is dense in the real number line. Under the hypothesis that $X$ is locally connected, Lipiński proved that $f$ is continuous if and only if $f$ is connected and satisfies pro-

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perty (G) Since the hypothesis of the local connectivety of $X$ was essential in his proof, he posed the problem: if every connected function on $X$ which satisfies property (G) is continuous, is $X$ locally connected ([9], P 639)? The purpose of this paper is to give a negative answer to Lipiński's question and to provide additional conditions on $X$ so that the question can be answered affirmatively.

Throughout, $X$ will denote a topological space and $R$ will denote the set of real numbers with the usual topology.

If $X$ is a connected denumerable space and if $f$ is any connected function on $X$, then $f$ is a constant and is thus continuous. If we can find a connected, non-locally connected denumerable space, we will have an example of a space where every connected function is continuous (and hence satisfies property (G)), but which is not locally connected. Such an example of a denumerable connected Hausdorff space is due to Bing [1]. It is easy to see this space is not locally connected.

Similar examples exist in spaces of higher cardinal. For instance, let $X$ be the set of real numbers with the topology generated by the open intervals of $R$ and the set $X = \{1/n : n = 1, 2, ...\}$. $X$ is Hausdorff and every connected function satisfying property (G) is continuous, but $X$ is not locally connected.

If, however, we assume that $X$ is a completely regular space (not necessarily $T_1$), then we have the following

THEOREM. *If $X$ is a completely regular space, then $X$ is locally connected if and only if every connected function on $X$ satisfying property (G) is continuous.*

Proof. Let $X$ be a completely regular space. If $X$ is locally connected, then every connected function satisfying property (G) is continuous (Lipiński [9]).

Conversely, assume $X$ is not locally connected. Then there is an open set $U \subset X$ and a component $D$ of $U$ such that $D$ is not an open subset of $U$. Let $y \in D \cap cl_X(U - D)$. Since $X$ is a completely regular space, there is a continuous function $f$ on $X$ to $[0,1] \subset R$ such that $f(x) = 1$ for $x \in X - U$ and $Z(f) = \{x \in X : f(x) = 0\}$ is a neighborhood of $y$. Also, there is a continuous function $g_0$ on $D$ to $[0,1]$ such that $g_0(y) = 1$ and $g_0(x) = 0$ for $x \in D - int_D Z(f)$. Let $g$ be an extension (not necessarily continuous) of $g_0$ to $X$ defined by $g(x) = g_0(x)$ for $x \in D$ and $g(x) = 0$ for $x \in X - D$. Then, the function $h = f + g$ is connected and satisfies property (G) but is not continuous.

To see that $h$ is not continuous, let $\{x_a\}$ be a net in $(U - D) \cap Z(f)$ that converges to $y$. Then $h(x_a) = 0$ for all $a$ but $h(y) = 1$.

For each $s \in S = R - \{0\}$, $h^{-1}(s) = f^{-1}(s) \cup g^{-1}(s)$ and both $f^{-1}(s)$ and $g^{-1}(s)$ are closed in $X$ so $h$ satisfies property (G).

If $E$ is a connected subset of $X$ and $E \cap D \cap Z(f) = \varnothing$, then $h[E] = f[E]$, which is connected. If $E \cap D \cap Z(f) \neq \varnothing$ and $E \subset U$, then $h[E] = (f|D+g_0)[E]$, which is connected. If $E \cap D \cap Z(f) \neq \varnothing$ and $E \cap X - U \neq \varnothing$, then for any real number $r$, $0 < r < 1$, there is an $x \epsilon E$ such that $f(x) = r$ (and thus $g(x) = 0$) and again $h[E]$ is connected.

Simple examples show that strengthening property (G) to the condition that $f^{-1}(r)$ is closed for all $r \epsilon R$ still is not sufficient for a connected function to be continuous in a non locally connected, completely regular space.

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