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ON A PROBLEM OF BANACH

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We shall prove that for each homogeneous Polish space there exists a continuous one-to-one map onto a compact metric space.

In "The Scottish Book" edited by R. Daniel Mauldin in 1981, the reader can find the following problem posed by Banach on July 17, 1935:

(a) When can a metric space [possibly of type (B)] be so metrized that it will become compact and so that all the sequences converging originally should also converge in the new metric?

(b) Can, e.g., the space c_0 be so metrized?

The problem was also published in Colloquium Mathematicum 1 (1947), p. 150. The part (a) can be formulated equivalently:

When can a metric space have a continuous one-to-one map onto a compact metric space?

We say that a metric space X has a *contraction to a compact metric space* Y iff there exists a continuous one-to-one map $f: X \to Y$ onto a compact metric space Y.

We shall show that each locally compact separable metric space has a contraction to a compact metric space. Also each Cartesian product of countably many spaces which have contractions to compact metric spaces has that property. In particular, the space ω^ω of irrational numbers and the countable infinite product $\boldsymbol{R}^\omega$ of lines have the property.

The first mathematicians to attack the Banach problem were Sikorski [9] and Katětov [2]. Sikorski proved that:

If $\{F_n: n \in \omega\}$ is a sequence of mutually disjoint closed subsets of a compact metric space Z such that $\delta(F_n) \to 0$, then the space

$$X = Z \setminus \bigcup_{n=0}^{\infty} F_n$$

has a contraction to a compact metric space.

Katětov solved the Banach problem for countable spaces. He proved that:

A countable regular space has a contraction to a compact metric space iff it is scattered (i.e., every nonempty subset has an isolated point; for countable metric spaces this is equivalent to the condition that the space does not contain a topological copy of the rational numbers).

It is known that the answer is "yes" to part (b) of the Banach problem and, in general, for Banach spaces. It follows from the famous Anderson–Kadec Theorem (see [1]):

All separable infinite-dimensional Fréchet spaces are homeomorphic to $\boldsymbol{R}^{\omega}$.

In his commentary, Mauldin writes ([7], p. 65) that part (a) of the Banach problem seems to be unsolved. He suggests the following restriction of the problem:

Let X be a Polish space (i.e., a complete separable metric space). Are there some simple conditions such that there is a continuous one-to-one map of X onto a compact metric space?

In [4] the author pointed out that there exists a 1-dimensional Polish space which does not have any contraction to a compact metric space and he proved that:

Each 0-dimensional complete metric space has a continuous one-to-one map onto a compact Hausdorff space.

From the above it follows that:

Each 0-dimensional Polish space has a contraction to a compact metric space.

Independently of Banach, a similar problem was raised by Russian mathematicians. They asked:

What kinds of spaces have continuous one-to-one maps onto Hausdorff compact spaces?

In 1939 Parkhomenko proved that if D is a countable subset of a compact metric space X, then the space $X \setminus D$ has a continuous one-to-one map onto a Hausdorff compact space. The result is valid for dyadic spaces (Belugin) and not valid for $\beta\omega \setminus \omega$ (Ponomarev). More information about the Russian problem can be found in Arkhangel'skii's appendix to the Russian second edition of Kelley's book [3].

The result of our note follows easily from [8]. But the aim of the note is to give a simple solution of the Banach problem using only classical theorems.

A space X is said to be *homogeneous* iff for each pair of points $x, y \in X$ there exists a homeomorphism h such that $h(x) = y$.

We shall prove:

THEOREM A. *Each homogeneous Polish space has a contraction to a compact metric space.*

The theorem implies immediately

COROLLARY. *If a topological group is a Polish space, then it has a contraction to a compact metric space.*

Let us start with simple observations.

OBSERVATION 1. *Let X be a locally compact separable metric space. Then X has a contraction to a compact metric space Y.*

Proof. Let Y be a topological space obtained by introducing a new topology on the set X in the following way. Choose a point $x_0 \in X$. Define a set $U \subset X$ to be open in Y iff U is open in X and $X \setminus U$ is compact whenever $x_0 \in U$. The space Y obtained in this way is compact and metrizable, and the identity map id: $X \to Y$, $\mathrm{id}(x) = x$, is continuous.

COROLLARY. *If*

$$X = \prod_{n=0}^{\infty} X_n$$

is a product of countably many locally compact separable metric spaces, then X has a contraction to a compact metric space.

Proof. Let $f_n \colon X_n \to Y_n$ be a continuous one-to-one map onto a compact metric space Y_n, $n \in \omega$. Then the map

$$f\colon \prod_{n=0}^{\infty} X_n \to \prod_{n=0}^{\infty} Y_n,$$

$$f(x_0, x_1, x_2, \ldots) := \langle f_0(x_0), f_1(x_1), f_2(x_2), \ldots \rangle$$

is continuous, one-to-one and onto.

A space X is *σ-compact* iff it is a union of countably many compact sets.

OBSERVATION 2. *If X is a homogeneous Polish space, then X is locally compact or X is non σ-compact.*

Proof. Assume that X is σ-compact,

$$X = \bigcup_{n=0}^{\infty} F_n,$$

where F_n is a compact subset of X. According to the Baire Theorem there is an $n \in \omega$ such that $\mathrm{int}\, F_n \neq \emptyset$. But this and homogeneity imply that X is locally compact.

In view of the above remarks, Theorem A will be a consequence of the following

THEOREM B. *Each non σ-compact Polish space has a continuous one-to-one map onto the product $\boldsymbol{R}^{\omega}$ of countably many lines.*

Theorem B is valid when we put I^{ω}, $I = [0, 1]$, instead of $\boldsymbol{R}^{\omega}$, but with $\boldsymbol{R}^{\omega}$ the theorem seems to look better because the assumption "X is not σ-compact" is necessary for the case $\boldsymbol{R}^{\omega}$ (because $\boldsymbol{R}^{\omega}$ is not σ-compact).

Now we prove the main result of this note. The idea of the proofs is taken from [8]. Fix some notation. For each $m \in \omega$ let us put

$$C_m = \boldsymbol{R}^{\omega} \times \boldsymbol{R}^{\omega}(m),$$

where

$$\boldsymbol{R}^{\omega}(m) = \{(x_0, x_1, x_2, \ldots) \in \boldsymbol{R}^{\omega}: x_n \neq m \text{ for finitely many } n \in \omega\},$$

$$\boldsymbol{R}^{\omega} = \prod_{i=0}^{\infty} \boldsymbol{R}_i, \qquad \boldsymbol{R}^{\omega} \times \boldsymbol{R}^{\omega} = \prod_{i=0}^{\infty} \boldsymbol{R}_i^1 \times \prod_{i=0}^{\infty} \boldsymbol{R}_i^2, \qquad \boldsymbol{R}_i = \boldsymbol{R}_i^{\alpha} = \boldsymbol{R},$$

and let

$$\pi_i: \boldsymbol{R}^{\omega} \to \boldsymbol{R}_i, \qquad \pi_i^{\alpha}: \boldsymbol{R}^{\omega} \times \boldsymbol{R}^{\omega} \to \boldsymbol{R}_i^{\alpha}$$

mean the projections.

LEMMA. *For a given $m \in \omega$ and for every continuous one-to-one map $f: M \to \boldsymbol{R}^{\omega} \times \boldsymbol{R}^{\omega} \setminus C_m$ from a closed subspace M of a metric separable space X, there exists a continuous one-to-one extension $f^*: X \to \boldsymbol{R}^{\omega} \times \boldsymbol{R}^{\omega}$ of the map f such that $f^*(X \setminus M) \subset C_m$.*

Proof. The set $X \setminus M$, being open, is an F_{σ}-set. So let

$$X \setminus M = \bigcup_{i=0}^{\infty} M_j,$$

where M_j is a closed subset of X and $M_j \subset M_{j+1}$ for each $j \in \omega$. Choose an embedding $h: X \hookrightarrow \boldsymbol{R}^{\omega}$ such that for each $n \in \omega$ the set $\{i \in \omega: \pi_n h = \pi_i h\}$ is infinite. Define continuous maps $f_j^{\alpha}: M \cup M_j \to \boldsymbol{R}_j^{\alpha}$ by

$$f_j^{\alpha}(x) = \begin{cases} \pi_j^{\alpha} f(x) & \text{if } x \in M,\ \alpha = 1, 2, \\ \pi_j h(x) & \text{if } x \in M_j,\ \alpha = 1, \\ m & \text{if } x \in M_j,\ \alpha = 2. \end{cases}$$

Let $(f_j^{\alpha})^*: X \to \boldsymbol{R}_j^{\alpha}$ be a continuous extension of the map f_j^{α}. The map $f^*: X \to \boldsymbol{R}^{\omega} \times \boldsymbol{R}^{\omega}$, defined by $\pi_j^{\alpha} f^* = (f_j^{\alpha})^*$, is an extension of the map f. To see that $f^*(X \setminus M) \subset C_m$ let us observe that if $x \in X \setminus M$, then there is an $n \in \omega$ such that $x \in M_i$ for each $i \geqslant n$. Consequently, $\pi_i^2 f^*(x) = m$ for each $i \geqslant n$, i.e., $f^*(x) \in C_m$. In order to show that f^* is one-to-one it suffices to verify that $f^*|X \setminus M$ is one-to-one, because we know that $f^*|M = f$ is one-to-one and $f^*(M) \cap f^*(X \setminus M) = \emptyset$. Notice that if x, $y \in X \setminus M$ and $x \neq y$, then there exists $i \in \omega$ such that $x, y \in M_i$ and $\pi_i h(x) \neq \pi_i h(y)$. But then

$$\pi_i^1 f^*(x) = \pi_i h(x) \neq \pi_i h(y) = \pi_i^1 f^*(y).$$

In the proof of the final result we need some knowledge about Borel sets. A metric space is said to be a *Borel space* (in accepted terminology: an *absolutely Borel set*) iff it is homeomorphic to a Borel set of a Polish space. The following facts will be useful:

1. If X is a Borel space and non σ-compact, then there exists a closed subset $\boldsymbol{P} \subset X$ homeomorphic to the space of irrational numbers (Hurewicz; for the proof see van Douwen's article in [5]).

2. A continuous one-to-one image of a Borel space is a Borel space (Suslin; see [6]).

3. Each Borel space is a continuous one-to-one image of a closed subspace of the space of irrational numbers (Lusin; see [6]).

THEOREM C. *Let X be a non σ-compact Borel space, $\boldsymbol{P} \subset X$ be a closed subspace homeomorphic to the space of irrational numbers, and let*

$$h: X \hookrightarrow \boldsymbol{R}^\omega \times \boldsymbol{R}^\omega(0)$$

be an embedding. Then for each open set $U \subset X$ such that $\boldsymbol{P} \subset U$ there exists a continuous one-to-one and onto map $f: X \to \boldsymbol{R}^\omega \times \boldsymbol{R}^\omega$ such that

$$f|X \setminus U = h|X \setminus U.$$

Proof. We may assume that

$$\boldsymbol{P} = \bigcup_{i=1}^{\infty} \boldsymbol{P}_i,$$

where $\boldsymbol{P}_i \cap \boldsymbol{P}_j = \emptyset$, $i \neq j$, and each set $\boldsymbol{P}_i$ is a closed subset of $\boldsymbol{P}$ homeomorphic to $\boldsymbol{P}$. Put $\boldsymbol{P}_0 = X \setminus U$. Choose a locally finite closed covering $\{F_i\colon i \in \omega\}$ of X such that

$$\boldsymbol{P}_i \subset F_i \setminus \bigcup_{j \neq i} F_j \quad \text{for each } i \in \omega.$$

For example, let

$$F_i = (g^*)^{-1}[i-\tfrac{1}{2}, i+\tfrac{1}{2}],$$

where $g^*\colon X \to (-\tfrac{1}{2}, \infty)$ is a continuous extension of the map

$$g\colon \bigcup_{i=0}^{\infty} \boldsymbol{P}_i \to (-\tfrac{1}{2}, \infty),$$

$g(x) = i$ whenever $i \in \boldsymbol{P}_i$.

Let us put

$$B_n := \boldsymbol{R}^\omega \times \boldsymbol{R}^\omega \setminus \bigcup_{j>n} C_j, \quad A_n := \bigcup_{i=0}^{n} F_i.$$

We have

$$B_n \subset B_{n+1}, \quad B_{n+1} = B_n \cup C_{n+1}, \quad \bigcup_{n=0}^{\infty} B_n = \mathbf{R}^{\omega} \times \mathbf{R}^{\omega},$$

$$A_n \subset A_{n+1}, \quad \bigcup_{n=0}^{\infty} A_n = X.$$

By induction we define continuous one-to-one maps

$$f_n\colon A_n \to \mathbf{R}^{\omega} \times \mathbf{R}^{\omega}$$

such that

$$f_{n+1}|A_n = f_n \quad \text{and} \quad B_n \subset f_{n+1}(A_{n+1}) \subset B_{n+1}.$$

Let $f_0 := h|A_0 = h|F_0$, $F_0 \supset X \setminus U$. Assume that the map f_n is defined. Let $f_{n+1}\colon A_{n+1} \to B_{n+1}$ be a continuous one-to-one extension of a map

$$f'_{n+1}\colon A_n \cup D_{n+1} \to B_n,$$

where D_{n+1} is a closed subset of P_{n+1} and the map

$$f'_{n+1}|D_{n+1} \to B_n \setminus f_n(A_n)$$

is one-to-one and onto (here we apply Suslin and Lusin Theorems and the Lemma).

Define $f\colon X \to \mathbf{R}^{\omega} \times \mathbf{R}^{\omega}$; $f(x) := f_n(x)$ iff $x \in A_n$. The map f is

1° continuous, because $f|F_n = f_n|F_n$ is continuous and the family $\{F_n\colon n \in \omega\}$ is locally finite;

2° one-to-one, because the maps are one-to-one and $A_n \subset A_{n+1}$, $\bigcup_{n=0}^{\infty} A_n = X$;

3° onto, because $B_n \subset f_{n+1}(A_{n+1})$ and $\bigcup_{n=0}^{\infty} B_n = \mathbf{R}^{\omega} \times \mathbf{R}^{\omega}$;

4° $f|X \setminus U = f_0|X \setminus U = h|X \setminus U$.

The proof of the theorem is completed.

Since $\mathbf{R}^{\omega} \times \mathbf{R}^{\omega}$ is homeomorphic to $\mathbf{R}^{\omega}$, and taking into account the theorem of Hurewicz, it is obvious that Theorem C is a generalization of Theorem B.

REFERENCES

[1] C. Bessaga and A. Pełczyński, *Selected Topics in Infinite-Dimensional Topology*, PWN – Polish Scientific Publishers, Warszawa 1975.

[2] M. Katětov, *On mappings of countable spaces*, Colloq. Math. 2 (1949), pp. 30–33.

[3] J. L. Kelley, *General Topology* (in Russian), Nauka, Moscow 1981.

[4] W. Kulpa, *A minimalization of 0-dimensional metric spaces*, Colloq. Math. 36 (1976), pp. 215–216.

[5] K. Kunen and J. E. Vaughan, *Handbook of Set-theoretic Topology*, North-Holland, Amsterdam 1984.

[6] K. Kuratowski, *Topology. I*, Academic Press, New York 1966.

[7] R. Daniel Mauldin (ed.), *The Scottish Book*, Birkhäuser, Boston 1981.

[8] E. G. Pytkeev, *The upper bounds of topologies* (in Russian), Mat. Zametki 20 (1976), pp. 489–500.

[9] R. Sikorski, *Remarks on a problem of Banach*, Colloq. Math. 1 (1948), pp. 285–288.

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