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PARTIALLY ORDERED GROUPS WITH TWO DISJOINT ELEMENTS

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Two elements $x > 0$ and $y > 0$ of a lattice ordered group G are said to be *disjoint*, if $x \wedge y = 0$. A set X of strictly positive elements of G is *disjoint* if any two elements $x_1, x_2 \in X$, $x_1 \neq x_2$, are disjoint. Conrad and Clifford [3] studied the structure of lattice ordered groups G satisfying the following condition:

(c_2) If $A \subset G$ is a disjoint set, then card $A \leqslant 2$.

A generalization of the results of [3] is given in Conrad's papers [4]-[6] (cf. also Fuchs [7], Chap. V, § 6).

Note that there does not exist a lattice ordered group containing exactly one pair of disjoint elements (since, if $x \neq y$ and $\{x, y\}$ is a disjoint set, then the set $\{2x, 2y\}$ is also disjoint and $\{2x, 2y\} \neq \{x, y\}$).

We can generalize the concept of disjointness for partially ordered groups as follows. Let P be a partially ordered set, $x, y \in P$ and let $U(x, y) \subset P$ be the set of all upper bounds of $\{x, y\}$. The set of all minimal elements of $U(x, y)$ will be denoted by $x \vee y$; the set $x \wedge y$ is defined dually (any of the sets $x \vee y$ and $x \wedge y$ may happen to be void). If for any $x, y \in P$ and any $v \in U(x, y)$ there exists $z \in x \vee y$ such that $z \leqslant v$ and if the dual condition also holds, then P is a *multilattice* (Benado [1]). A partially ordered group G for which the corresponding partially ordered set $(G; \leqslant)$ is a multilattice is called a *multilattice group*; such groups were considered by McAllister [8]. Now let G be any partially ordered group. A subset of strictly positive elements of G will be called *disjoint*, if $0 \in x \wedge y$ for any two elements $x, y \in X$, $x \neq y$; such elements x and y are called *disjoint*. If $x, y \in X$ and neither $x \leqslant y$ nor $y \leqslant x$, we call x and y *incomparable* and write $x \,|\, y$.

In this note there are studied partially ordered groups containing exactly one pair of disjoint elements. In other words, we will consider partially ordered groups G having the property

(q_2) There exist disjoint elemets $x, y \in G$ such that if $A \subset G$ is a disjoint subset and card $A > 1$, then $A = \{x, y\}$.

In statements 1-15 we assume that G satisfies (q_2). Let $a, b \in G$, $a \leqslant b$. The *interval* $[a, b]$ is the set of all $c \in G$ such that $a \leqslant c \leqslant b$. The interval $[a, b]$ is *prime* if $[a, b] = \{a, b\} \neq \{a\}$.

1. *Intervals* $[0, x]$ *and* $[0, y]$ *are prime.*

Proof. Let $0 \neq x_1 \in [0, x]$ and $0 \neq y_1 \in [0, y]$. Then $0 \in x_1 \wedge y_1$, whence $\{x_1, y_1\} = \{x, y\}$ according to (q_2). If $x_1 = y$, then $y < x$, $x \wedge y = \{y\}$, a contradiction. Therefore $x_1 = x$ and analogously $y_1 = y$.

2. *Interval* $[nx, (n+1)x]$ *is prime for any integer* n.

Proof. From the definition of a partially ordered group it follows that $[0, x] \sim [nx, (n+1)x]$, where the symbol $\sim$ denotes an isomorphism with regard to the partial order; our assertion is now implied by statement 1.

3. *Interval* $[x, x+y]$ *is prime and* $x+y \in x \vee y$.

Proof. Since $[0, y] \sim [x, x+y]$, the first assertion follows from statement 1. This, in turn, implies $x+y \in x \vee y$.

4. $2x \in x \vee y$.

Proof. Since $0 < 2x$ and $[0, y]$ is a prime interval, we have either $0 \in 2x \wedge y$ or $2x > y$. But $2x > x$ and $2x \neq y$ (since $2x = y$ implies $x \wedge y = \{x\}$, a contradiction), whence $\{2x, y\} \neq \{x, y\}$ and therefore, by (q_2), $0 \notin 2x \wedge y$; thus $2x > y$. Moreover, since the interval $[x, 2x]$ is prime, we get $2x \in x \vee y$.

4.1. Remark. Obviously, we can interchange x and y in statements 2, 3 and 4.

The mapping $\varphi(t) = -t$ $(t \in G)$ is a dual automorphism of a partially ordered set G; hence and from (q_2) it follows that

5. $0 \in (-x) \vee (-y)$.

Indeed, if $a, b \in G$, $a < 0$, $b < 0$, $0 \in a \vee b$, then $\{a, b\} = \{-x, -y\}$.

6. $2x = 2y$.

Proof. According to 4 we have $y < 2x$. Moreover, again from 4, we get $0 \in (-x) \vee (y-2x)$, and since $-x < 0$ and $y-2x < 0$, by 5 we have $\{-x, y-2x\} = \{-x, -y\}$. Consequently, $y-2x = -y$, whence $2y = 2x$.

7. *Intervals* $[y-x, x]$ *and* $[y-x, y]$ *are prime and* $y-x \in x \wedge y$.

Proof. We have $[y-x, x] \sim [y, 2x] = [y, 2y]$ and the last interval is prime in view of 2. Furthermore, $[y-x, y] \sim [-x, 0]$ and the interval $[-x, 0]$ is dually isomorphic to $[0, x]$, whence by 1 the interval $[-x, 0]$ is prime. The last assertion is an immediate consequence of the preceding.

8. $y+x = x+y$.

Proof. By statement 3 and remark 4.1 intervals $[x, y+x]$ and $[y, y+x]$ are prime and $y+x \in x \vee y$. Hence and from 3 it follows (according to the definition of the set $x \vee y$) that either $x+y = y+x$ or $x+y \mid y+x$. If $x+y$ and $y+x$ are incomparable, then $x \in (x+y) \wedge (y+x)$, whence $0 \in y \wedge (-x+y+x)$, and thus, by (q_2), $-x+y+x = x$, $y = x$, a contradiction. Therefore $x+y = y+x$.

Let H be the subgroup of G generated by the set $\{x, y\}$. From 8 we get as a corollary:

9. *The subgroup H is abelian.*

From 6 and 9 it follows that

10. *If $y-x = t$, then $2t = 0$.*

11. *Any $z \in H$ can be uniquely expressed in the form $z = mx+nt$, where m is an integer and $n \in \{0, 1\}$.*

Proof. Let $z \in H$. According to 9 there exist integers s_1 and s_2 such that $z = s_1x+s_2y$. Thus $z = mx+s_2t$, where $m = s_1+s_2$. If $s_2 = 2k$ ($s_2 = 2k+1$), then, by 10, $s_2t = nt$ with $n = 0$ ($n = 1$). Assume that $mx+nt = 0$. If $n = 0$, then $mx = 0$, whence $m = 0$. Let $n \neq 0$; then $n = 1$ and, consequently, $mx = -t = t$. Elements mx and 0 are comparable and $t \mid 0$, a contradiction. Hence $mx+nt = 0$ implies $m = 0$ and $n = 0$, and the considered expression is unique.

12. $mx+nt > 0 \Leftrightarrow m > 0$.

Proof. According to 10 we can suppose that $n \in \{0, 1\}$. Let $n = 0$; obviously, $mx > 0$ if and only if $m > 0$. Further, let $n = 1$. Then $mx+t = mx-t = (m+1)x-y$. If $m > 0$, then $m+1 \geqslant 2$, whence $(m+1)x \geqslant 2x = 2y > y$ and $(m+1)x-y > 0$, and, consequently, $mx+t > 0$. If $m = 0$, then $mx+t \mid 0$. In the case of $m < 0$, we have $-mx+t > 0$, whence $mx+t < 0$.

13. *Let H_1 be the set of all pairs (m, n), where m is an integer and $n \in \{0, 1\}$. We define in H_1 the operation $+$ componentwise, n_1+n_2 being taken* mod 2. *For (m_1, n_1), $(m_2, n_2) \in H_1$ we put $(m_1, n_1) < (m_2, n_2)$ if $m_1 < m_2$. Then H_1 is a partially ordered group isomorphic to the partially ordered group H.*

This follows from 9, 10, 11 and 12. It is easy to see that H_1 satisfies (q_2).

13.1. A multilattice M is said to be *transitive* if it satisfies the following condition: for any $a_i, b_i, c_i \in M$ $(i = 1, 2)$ such that $a_1 \in a_2 \vee b_1$, $b_2 \in a_2 \wedge b_1$, $b_1 \in b_2 \vee c_1$, $c_2 \in b_2 \wedge c_1$, and $c_1 \nleqslant a_2$, the relations $a_1 \in a_2 \vee c_1$ and $c_2 \in a_2 \wedge c_1$ hold true. $(H_1; \leqslant)$ is an example of a transitive multilattice (cf. Benado [2]). The partially ordered group H_1 shows that there exist transitive multilattice groups that are not lattice ordered (this answers a question of Benado ([2], Problem 6)).

14. *H is a convex subset of the partially ordered set $(G, \leqslant)$.*

Proof. Let $0 < v < z$, $z \in H$, $v \in G$. Then there exists a positive integer m such that $0 < v < mx$; let m be the minimal positive integer with this property. It follows from 1 that $m > 1$. Assume that $v \notin H$. Then $v \mid (m-1)x$, since $[(m-1)x, mx]$ is a prime interval. Moreover, $mx \in (m-1)x \vee v$, whence $0 \in (-x) \vee (v-mx)$. According to 5 this implies $v-mx = -y$, and thus $v \in H$, a contradiction.

15. *If* $v \in G, v \notin H, v > 0$, *then* $v > z$ *for each* $z \in H$.

Proof. Assume that there exists $z \in H$ such that $v \ngtr z$. Then there exists a minimal positive integer m with the property $mx \nless v$. By 14, $mx \mid v$ holds and thus by 2 we have $(m-1)x \in mx \wedge v$, whence $0 \in x \wedge \wedge \big(v-(m-1)x\big)$ which implies $v-(m-1)x = y$, $v \in H$, a contradiction.

In the same way we can prove an analogical statement for $v < 0$. The previous results can be summarized as follows:

16. THEOREM. *Let* G *be a partially ordered group fulfilling* (q_2). *Then there exists a convex subgroup* H *of* G *isomorphic to the partially ordered group* H_1 *from* 13. *For any* $v \in G \setminus H, v > 0$ $(v < 0)$ *and any* $z \in H$ *the relation* $z < v$ $(v < z)$ *holds.*

An element $a \in G$ is said to be *archimedean*, if the set $\{na\}$ $(n = 0, \pm 1, \pm 2, \ldots)$ is not bounded in G. Let us consider the following condition for G:

($\overline{q}_2$) G satisfies (q_2) and at least one of the elements x, y is archimedean.

17. *Let* G *be a directed partially ordered group. Then* G *fulfils* ($\overline{q}_2$) *if and only if* G *is isomorphic to* H_1.

Proof. Obviously, H_1, is directed and satisfies ($\overline{q}_2$). Assume that G is directed and fulfils ($\overline{q}_2$). Let $w \in G$. Since G is directed, there exist elements $u, v \in G$ such that $u < 0 < v$ and $u < w < v$. We can suppose that x is archimedean. Then it follows from 15 that u and v belong to H; thus, by 14, w belongs to H as well. Hence $G = H$ and it follows from 13 that G and H_1 are isomorphic.

18. *Let* G *be a directed multilattice group satisfying* (q_2). *If* $w \in G$ *and* $w \mid 0$, *then* $w \in H$.

Proof. Let $w \in G$ and $w \mid 0$. There exists then $u \in G$ such that $u < 0$ and $u < w$. Since $(G, \leqslant)$ is a multilattice, there exists $u_1 \in 0 \wedge w$ with the property $u_1 \geqslant u$. Hence $0 \in (-u_1) \wedge (w-u_1)$ and thus, according to (q_2), $\{-u_1, w-u_1\} = \{x, y\}$. If $-u_1 = x$ and $w-u_1 = y$, then $w \in H$; the case $-u_1 = y, w-u_1 = x$ is analogous.

19. *If* G *satisfies* (q_2), *then the subgroup* H (cf. 9) *is normal.*

Proof. Let $a \in G$. The mapping $z \to \varphi(z) = -a+z+a$ is an automorphism of a partially ordered set $(G; \leqslant)$ and $\varphi(0) = 0$. Therefore, $0 \in \varphi(x) \wedge \varphi(y)$. By ($q_2$), $\{\varphi(x), \varphi(y)\} = \{x, y\}$. Since H is a subgroup generated by $\{x, y\}$, $\varphi(H)$ is a subgroup of G generated by $\varphi(x)$ and $\varphi(y)$; hence $\varphi(H) = H$.

Let A be a normal convex subgroup of a partially group G. If for each $v \in G, v \notin A$, either $v > 0$ or $v < 0$ holds, then G is said to be a *lex-*

extension of the partially ordered group A (cf. Conrad [4] and [5]). In such a case G/A is linearly ordered. If $c, d \in G$, $c+A \neq d+A$, $c_1 \in c+A$, $d_1 \in d+A$, and $c < d$, then $c_1 < d_1$. Indeed, $c_1 - d_1 \notin A$, and so the elements $c_1 - d_1$ and 0 are comparable. If $c_1 - d_1 > 0$, then $c_1 > d_1$, thus in the partially ordered group G/A we have $d+A = d_1+A < c_1+A = c+A < d+A$, a contradiction. Hence $c_1 - d_1 < 0$, i.e. $c_1 < d_1$.

20. THEOREM. *Let G be a directed multilattice group. Then G satisfies* (q_2) *if and only if G is a lex-extension of a partially ordered group isomorphic to H_1.*

Proof. Assume that G is a lex-extension of a partially ordered group A isomorphic to H_1. Let $c, d \in G, c > 0, d > 0, 0 \in c \wedge d$. If $c+A \neq d+A$, then either $c-d > 0$ or $c-d < 0$, whence $c \wedge d = \{d\}$ or $c \wedge d = \{c\}$, a contradiction. If $c+A = d+A \neq A$, then $a < c$ and $a < d$ for each $a \in A$, whence $0 \notin c \wedge d$. Therefore $\{c, d\} \subset A$ and thus, since A satisfies (q_2), the partially ordered group G does it. Conversely, let us suppose that G satisfies (q_2). According to 16 and 19, H is a normal convex subgroup of G and by 18, for any $v \in G$, $v \notin H$, either $v > 0$ or $v < 0$ holds. Hence G is a lex-extension of H.

21. Let A be a subgroup of a partially ordered group G fulfilling the following conditions:

(a) A is a convex subset of $(G; \leqslant)$;

(b) A is a normal subgroup of the group $(G; +)$;

(c) if $c, d \in G, c+A \neq d+A, c_1 \in c+A, d_1 \in d+A$, and $c < d$, then $c_1 < d_1$.

Under these assumptions G will be said to be a *generalized lex-extension* of A.

Remark. It is easy to prove that a generalized lex-extension G of A is a lex-extension of A if and only if G/A is linearly ordered.

22. *Let G be a generalized lex-extension of a directed group $A \neq \{0\}$. If $c, d \in G$, $c \| d$ and $c \wedge d \neq \emptyset$, then $c+A = d+A$.*

Proof. Assume that $c, d \in G$, $c \| d$, $c+A \neq d+A$ and $e \in c \wedge d$. Then $e < c$ and $e < d$. If $e+A = c+A$, we would have, by 21 (c), $c < d$, a contradiction; hence $e+A \neq c+A$ and, analogously, $e+A \neq d+A$. By 21 (c) we then have $e_1 < c$ and $e_1 < d$ for each $e_1 \in e+A$. There exists $a \in A$, $a > 0$; if we put $e_1 = e+a$, then $e < e_1 \in e+A$ and this shows that $e \notin c \wedge d$, a contradiction.

23. *Let G be a generalized lex-extension of a directed group $A \neq \{0\}$. Then G satisfies* (q_2) *if and only if A does.*

Proof. Let G satisfy (q_2). Then $0 \in x \wedge y$ and $x \| y$. According to 22, $x+A = y+A$. If $x+A \neq A$, then by 21 (c) we have $x > a$ and $y > a$ for each $a \in A$, whence $0 \notin x \wedge y$, a contradiction. Therefore $x, y \in A$ and thus A satisfies (q_2). Conversely. assume that A fulfills (q_2) and let $c, d \in G$, $c \| d$, $0 \in c \wedge d$. Then $c, d \in A$, whence $\{c, d\} = \{x, y\}$ and thus G also satisfies (q_2).

24. *If G satisfies* (q_2), *then G is a generalized lex-extension of H.*

Proof. According to 14 and 19 it remains to verify condition 21(c) only. Let $c, d \in G$, $c+H \neq d+H$, $c < d$, $c_1 \in c+H$, $d_1 \in d+H$. Since there exist elements $h_1, h_2 \in H$ such that $c_1 = c+h_1$ and $d_1 = d+h_2$, it suffices to prove that $d > c+h$ for any $h \in H$. For each $h \in H$ there exists a positive integer m such that $h < mx$; thus we have to prove that $d > c+mx$ for each positive integer m. Assume that there exists a positive integer m satisfying $d \not> c+mx$ and take the least m with this property. If $d < c+mx$, then by the convexity of the set $c+H$ we get $d \in c+H$, a contradiction. Hence $d \| c+mx$, $d > c+(m-1)x$. Since $[c+(m-1)x, c+mx]$ is a prime interval, we have $c+(m-1)x \in d \wedge (c+mx)$. This implies $0 \in (d-(m-1)x-c) \wedge (c+x-c)$, thus $d-(m-1)x-c$ is by (q_2) equal to x or y and therefore $d+H = c+H$, a contradiction. This completes the proof.

25. THEOREM. *A partially ordered group G satisfies* (q_2) *if and only if it is a generalized lex-extension of a partially ordered group isomorphic to H_1.*

This follows from 13, 23 and 24.

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