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CONCERNING IRREDUCIBLE CONTINUA OF HIGHER DIMENSION

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Let $\mathscr{K}$ denote the class of all compact metric continua K such that there exists an upper semi-continuous decomposition G of a compact metric irreducible continuum M with each element of G homeomorphic to K and with decomposition space M/G an arc. Knaster [2] showed that an arc is in $\mathscr{K}$. In [4] it is shown that the arc is the only connected finite 1-polyhedron in $\mathscr{K}$. Also in that paper, the question is raised of whether the 2-cell is in $\mathscr{K}$. In [1] it is shown that if n is a positive integer, there exists an n-dimensional continuum in $\mathscr{K}$ and that the Hilbert cube is in $\mathscr{K}$. In this paper it is shown that if n is a positive integer, then the n-cell is in $\mathscr{K}$.

THEOREM 1. *There exists a compact metric continuum M such that*

(1) *M is irreducible*;

(2) *there exists an upper semi-continuous collection G of arcs filling up M such that M/G is an arc*; *and*

(3) *there exists a countable subcollection H of G such that*

(a) *if h is in H, h contains an arc Z_h such that each point of Z_h is a separating point of M and Z_h contains every separating point of M in h,*

(b) *if S denotes the set of all points P such that P is an end-point of Z_h, for some h in H, then S is dense in $M - \bigcup_{h\in H} Z_h$,*

(c) *if $\varepsilon > 0$, then only finitely many members h of H have $d(Z_h) > \varepsilon$ and*

(d) *$\bigcup_{h\in H} Z_h$ contains all separating points of M.*

Proof. Let

$$g_1(x) = \begin{cases} \sin\dfrac{1}{x} & \text{for } 0 < x \leqslant \dfrac{1}{\pi}, \\[2ex] \sin\dfrac{1}{x-2/\pi} & \text{for } \dfrac{1}{\pi} \leqslant x < \dfrac{2}{\pi}, \end{cases}$$

and

$$g_2(x) = g_1(x) - \frac{1}{4}.$$

Denote by g_1 and g_2 the graphs of the functions defined above. Let A denote the vertical interval from $(0, -5/4)$ to $(0, 1)$ and B denote the vertical interval from $(2/\pi, -5/4)$ to $(2/\pi, 1)$. Let H_1 denote $A \cup B \cup g_1 \cup g_2$ and I_1 denote the interior of $A \cup B \cup g_1 \cup g_2$. Let a_1 denote a countable sequence of mutually exclusive arcs such that if u is in a_1, then

(1) u lies in $H_1 \cup I_1$,

(2) the endpoints of u both lie on g_1 or g_2,

(3) u intersects g_1 and g_2 and $u \cap (g_1 \cup g_2)$ is the set consisting of the endpoints of u together with an arc Z_u,

(4) the diameter of each component of $u - u \cap (g_1 \cup g_2)$ is greater than 1,

(5) if $d(Z_u)$ denotes the diameter of Z_u for each u in a_1, then $\sum_{u \in a_1} d(Z_u) \leqslant \frac{1}{2}$,

(6) if S_1 denotes the set of all points P such that P is an endpoint of Z_u for some u in a_1, then the limiting set of S_1 is $A \cup B$.

If u is an element of a_1, let D_u denote the component of $(H_1 \cup I_1) - u$ that does not contain A or B. Let L_1 denote $(H_1 \cup I_1) - \bigcup_{u \in a_1} D_u$. Let C_1 denote the family of the components of $L_1 - (a_1^* \cup A \cup B)$. Let $c_{11}, c_{12}, c_{13}, \ldots$ denote the elements of C_1. If c is an element of C_1, let A_c and B_c denote the components of $\bar{c} \cap (a_1^*)$ and let g_{1c} and g_{2c} denote the components of $\bar{c} \cap (g_1 \cup g_2)$. Let x and y denote the intervals $[(0, 1), (2/\pi, 1)]$ and $[(0, -5/4), (2/\pi, -5/4)]$, respectively.

Let $f_{c_{11}}, f_{c_{12}}, \ldots$ denote a sequence such that for each i, $f_{c_{1i}}$ denotes a homeomorphism from the square disc bounded by $A \cup B \cup x \cup y$ onto $\bar{c}_{1i}$ such that

(1) $f_{c_{1i}}(A \cup B) = A_{c_{1i}} \cup B_{c_{1i}}$,

(2) $f_{c_{1i}}(x \cup y) = g_{1c_{1i}} \cup g_{2c_{1i}}$,

(3) if u is in a_1, the diameter of each component of $f_{c_{1i}}[u - u \cap (g_1 \cup g_2)]$ is greater than 1,

(4) the area of $L_2 = A \cup B \cup a_1^* \cup \bigcup_{i>0} f_{c_{1i}}(L_1)$ is less than one half the area of L_1, and

(5) $\sum_{i>0} \sum_{u \in a_1} d[f_{c_{1i}}(Z_u)] \leqslant \frac{1}{4}$.

Continuing inductively let a_n denote the collection of arcs to which v belongs if and only if for some element $c_{n-1,i}$ of C_{n-1}, v is $f_{c_{n-1,i}}(u)$ for some element u of a_1. Let C_n denote the family of the components of $L_n - (a_1^* \cup a_2^* \cup \ldots a_n^* \cup A \cup B)$. If c is an element of C_n, let A_c and B_c denote the components of $\bar{c} \cap a_n^*$ and g_{1c} and g_{2c} denote the closures of the components of $B(\bar{c}) - (A_c \cup B_c)$, where $B(\bar{c})$ is the boundary of $\bar{c}$. Let $f_{c_{n1}}, f_{c_{n2}}, \ldots$

denote a sequence such that for each $i, f_{c_{ni}}$ denotes a homeomorphism from the square disc bounded by $A \cup B \cup x \cup y$ onto $\bar{c}_{ni}$ such that

(1) $f_{c_{ni}}(A \cup B) = A_{c_{ni}} \cup B_{c_{ni}}$,

(2) $f_{c_{ni}}(x \cup y) = g_{1c_{ni}} \cup g_{2c_{ni}}$,

(3) if u is in a_1, the diameter of each component of $f_{c_{n,i}}(u - u \cap (g_1 \cup g_2))$ is greater than 1,

(4) the area of $L_{n+1} = A \cup B \cup \bigcup_{i>0} f_{c_{ni}}(L_1) \cup a_1^* \cup a_2^* \cup \ldots \cup a_n^*$ is less than $A(L_1)/(n+1)$, where $A(L_1)$ is the area of L_1, and

(5) $\sum_{i>0} \sum_{u \epsilon a_1} d[f_{c_{ni}}(Z_u)] \leqslant 1/2^n$.

$L_1, L_2, L_3, \ldots$ is a monotone sequence of compact continua and the common part L of all of them is an irreducible continuum, since the set of all points of L which separate A from B in L is dense in L. The collection to which h belongs if and only if for some u of a_1, some n, and some i, h is $f_{c_{ni}}(Z_u)$ is a countable collection of mutually exclusive arcs satisfying the condition of the conclusion of the Theorem.

Let K denote the collection to which g belongs if and only if (1) g is a point of $(A \cup B \cup a_1^* \cup a_2^* \cup \ldots)$ or (2) for some component c of $L - (A \cup B \cup \bigcup_{n=1}^{\infty} a_n^*)$ and some horizontal line l intersecting c, g is the set of all points of c on l. K is an upper semicontinuous collection of mutually exclusive closed point sets filling up L. Let M denote L/K. Let G denote the collection to which g belongs if and only if (1) g is A, B, or an element of a_n for some n, or (2) g is a component of $M - (A \cup B \cup \bigcup_{n=1}^{\infty} a_n^*)$. M is an irreducible continuum from A to B, and M/G is an arc. Furthermore, each element of G is an arc. Also, M is chainable and therefore embeddable in the plane. It can be seen that M satisfies all the conditions of the conclusion of the Theorem by letting $H = \bigcup_{n=1}^{\infty} a_n$ and if h is in a_n, $Z_h = f_{c_{n-1,i}}(Z_u)$, where u is in a_1 and i is a positive integer such that $h = f_{c_{n-1,i}}(u)$.

THEOREM 2. *The 2-cell is in $\mathscr{K}$.*

Proof. Let M denote a compact continuum in the plane that satisfies the conditions of Theorem 1 and let G, $H = \{h_1, h_2, h_3, \ldots\}$ and $Z_{h_1}, Z_{h_2}, Z_{h_3}, \ldots$ be as described in Theorem 1.

Let K denote the collection to which k belongs if and only if for some positive integer i, k is the closure of a component of $h_i - Z_{h_i}$. Now

$$(G-H)^* \cup K^* = \overline{M - \bigcup_{i=1}^{\infty} Z_{h_i}}$$

and $(G-H) \cup K$ is an upper semi-continuous collection of mutually exclusive arcs filling up $\overline{M - \bigcup_{i=1}^{\infty} Z_{h_i}}$.

Let $g_1, g_2, g_3, \ldots$ denote a countable collection of subintervals of $[0, 1]$ such that

(1) $\lim_{i\to\infty} d(g_i) = 0$,

(2) $\bigcup_{i=1}^{\infty} g_i \times Z_{h_i}$ is dense in

$$M' = \{\bigcup_{i=1}^{\infty} g_i \times Z_{h_i}\} \cup \{[0, 1] \times [(G-H) \cup K]^*\}.$$

This is clearly possible since $\bigcup_{i=1}^{\infty} Z_{h_i}$ is dense in M. Let U denote the collection to which u belongs if and only if U is a point of $M' - \bigcup_{i=1}^{\infty} g_i \times Z_{h_i}$ or, for some positive integer n and some point P of g_n, u is $P \times Z_n$. U is an upper semi-continuous decomosition of M' since $\lim_{i\to\infty} d(g_i \times Z_{h_i}) = 0$.

Let A and B denote the end elements of M/G. M'/U is an irreducible continuum from $([0, 1] \times A)/U$ to $([0, 1] \times B)/U$, since if P is a point of M'/U and R is a domain containing P, then there exists a positive integer i such that R contains $(g_i \times Z_{h_i})/U$. Let U' denote the collection to which u' belongs if and only if

(1) for some element g of $G-H$, u' is $\big[[0, 1] \times g\big]/U$ or

(2) for some i, u' is $\{[0, 1] \times (\overline{h_i - Z_{h_i}}) \cup (g_i \times Z_{h_i})\}/U$.

U' is an upper semi-continuous collection of mutually exclusive 2-cells filling up M'/U such that U' is an arc with respect to its elements.

THEOREM 3. *If n is a positive integer, the n-cell is in $\mathscr{K}$.*

Proof. Since two n-cells identified on $(n-1)$-cells of faces yield an n-cell, an inductive argument entirely analogous to that of Theorem 2 suffices for the proof.

Remarks. Since by [4] the simple closed curve does not belong to $\mathscr{K}$, one might wonder if the boundary of a 3-cell belongs to $\mathscr{K}$ (**P 865**). Also, does every member of $\mathscr{K}$ contain disjoint copies of itself? (**P 866**)

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