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THE CURVATURES OF CERTAIN SURFACES OF TRANSLATION

BY

D. KOUTROUFIOTIS AND B. WEGNER (BERLIN)

This note* concerns the curvature diagram of certain complete surfaces in the Euclidean 3-space E^3. We shall namely prove the following

THEOREM. *Let F be a complete C^2-surface in E^3 permitting a global Cartesian representation $z(x, y) = f(x)+g(y)$. Let $\varkappa_1$ and $\varkappa_2$ be its principal curvatures. Then*

$$\inf\{(\varkappa_1^2+\varkappa_2^2)(P)\,|\,P\in F\} = 0.$$

We note that a surface of the above type is called a *surface of translation* (S. Lie), since it can be generated by moving the space curve $z(t) = f(t)$, $x(t) = t$, $y(t) = 0$ rigidly and parallel to itself along the space curve $z(t) = f(0)+g(t)$, $x(t) = 0$, $y(t) = t$.

First we prove a lemma of independent interest.

LEMMA. *Let ψ: $[a, b) \to \boldsymbol{R}^+$ be a monotonically increasing C^1-function satisfying $d\psi/dx \geqslant c\psi^3$ for a certain constant $c > 0$ and $\psi(a) > 0$. Then b is a finite number and the possibly improper integral $\int_a^b \psi(x)dx$ converges to a finite number.*

Proof. Since $\psi' \geqslant c\psi^3$ and ψ is monotonically increasing, we have

$$\psi'(x) \geqslant c\psi^3(a) > 0.$$

Using the mean value theorem we now obtain

$$\psi(x)-\psi(a) \geqslant c\psi^3(a)(x-a).$$

In case where ψ is bounded, we deduce from the last inequality, by letting x tend to b, that

$$b < \infty \quad \text{and} \quad \int_a^b \psi(x)dx < \infty.$$

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Consider next the case in which ψ tends to infinity if x tends to b. Now $\psi' > 0$, so that ψ is strictly increasing. It follows that then there exists a positive integer n_0 such that for all integers $n \geqslant n_0$ the equation $\psi(x) = n$ has a unique solution a_n. By the mean value theorem again, for a certain $\xi_n \in (a_n, a_{n+1})$ we have

$$\frac{1}{a_{n+1}-a_n} = \frac{\psi(a_{n+1})-\psi(a_n)}{a_{n+1}-a_n} = \psi'(\xi_n) \geqslant c\psi^3(\xi_n) \geqslant c\psi^3(a_n) = cn^3.$$

Thus

$$a_{m+1}-a = a_{n_0}-a+\sum_{n=n_0}^{m}(a_{n+1}-a_n) \leqslant a_{n_0}-a+\frac{1}{c}\sum_{i=1}^{\infty}\frac{1}{i^3},$$

whence

$$a_{m+1} \leqslant a_{n_0}+\frac{1}{c}\sum_{i=1}^{\infty}\frac{1}{i^3}$$

for all integers $m \geqslant n_0$. The right-hand side of the last inequality is a finite number independent of m. Now $a_{m+1} \to b$ as $m \to \infty$, therefore $b < \infty$. Furthermore, for all $m \geqslant n_0$ we have

$$\begin{aligned}\int_a^{a_{m+1}} \psi\,dx &= \int_a^{a_{n_0}} \psi\,dx+\sum_{n=n_0}^{m}\int_{a_n}^{a_{n+1}} \psi\,dx\\ &\leqslant \int_a^{a_{n_0}} \psi\,dx+\sum_{n=n_0}^{m}\psi(a_{n+1})(a_{n+1}-a_n) \leqslant \int_a^{a_{n_0}} \psi\,dx+\frac{1}{c}\sum_{i=1}^{\infty}\frac{i+1}{i^3}.\end{aligned}$$

This last infinite series is again convergent, therefore

$$\int_a^b \psi\,dx = \lim_{m\to\infty}\int_a^{a_{m+1}} \psi\,dx < \infty,$$

since ψ is monotonically increasing, q.e.d.

Proof of the Theorem. Recall that a surface F is *complete* if every rectifiable, divergent ray on it has infinite length; by a *divergent ray* we mean a continuous mapping $\varphi\colon [a, b) \to F$ with the property that $\varphi([a, b))$ does not lie in any compact subset of F.

Setting $f_x = df/dx$ etc. we compute the Gaussian curvature of F and the mean one,

$$K(x, y) = (1+f_x^2+g_y^2)^{-2}f_{xx}g_{yy},$$

$$H(x, y) = \tfrac{1}{2}(1+f_x^2+g_y^2)^{-3/2}[(1+f_x^2)g_{yy}+(1+g_y^2)f_{xx}],$$

and

$$
(1) \quad (\varkappa_1^2+\varkappa_2^2)(x, y) = 2(2H^2-K)(x, y)
$$
$$
= (1+f_x^2+g_y^2)^{-3}[(1+f_x^2)^2 g_{yy}^2+2f_x^2 g_y^2 f_{xx} g_{yy}+(1+g_y^2)^2 f_{xx}^2]
$$

We now proceed indirectly. Assume that $\varkappa_1^2+\varkappa_2^2 \geqslant c^2 > 0$ for some constant $c > 0$. Let f be defined on the interval (a, b), where a and b are in $R \cup \{-\infty, +\infty\}$. From (1) it follows easily that

$$
(2) \qquad c^2 \leqslant (|f_{xx}|+|g_{yy}|)^2 .
$$

From (2) we infer that either $|g_{yy}| \geqslant c/2$ or $|f_{xx}| \geqslant c/2$. Indeed, if for some y_0 we have $|g_{yy}(y_0)| < c/2$, then for all $x \in (a, b)$ we must have $|f_{xx}| > c/2$, since f does not depend on y. Say, we have $|f_{xx}| \geqslant c/2$ throughout. Performing a reflection, if need, we may assume without loss of generality that $f_{xx} \geqslant c/2 > 0$. This implies that f_x is strictly increasing. We assert that f_x tends to $+\infty$ as $\tilde{x}$ tends to b; if namely $b = +\infty$, then

$$
f_x(\tilde{x})-f_x(x_0) = f_{xx}(\xi)(\tilde{x}-x_0) \geqslant \frac{c}{2}(\tilde{x}-x_0)
$$

for a fixed $x_0 \in (a, b)$ and a certain ξ between $\tilde{x}$ and x_0. Therefore, $f_x(\tilde{x}) \to +\infty$ as $\tilde{x} \to b$. If, on the other hand, $b < \infty$, then the assertion follows from the observation that the divergent ray l,

$$
l = \{x(t) = t, y(t) = y_0, z(t) = f(t)+g(y_0) \mid x_0 \leqslant t < b\},
$$

must have length

$$
L(l) = \int_{x_0}^{b} (1+f_x^2)^{1/2} dx = +\infty,
$$

since F is complete.

Consider a fixed $y = y_0$ and set $g_y(y_0) = c_1$, $|g_{yy}(y_0)| = c_2$. From (1) we infer that

$$
c^2 \leqslant (1+f_x^2)^{-3}[(1+f_x^2)^2 c_2^2+(1+c_1^2)^2 f_{xx}^2+2c_1^2 c_2 f_x^2 f_{xx}].
$$

Since

$$
\lim_{x \to b} \frac{c_2^2}{1+f_x^2} = 0,
$$

for appropriate numbers $a_1 \in (a, b)$ and $c_3 > 0$ we have

$$
c_3 \leqslant \frac{f_{xx}+Af_x^2}{(1+f_x^2)^{3/2}} \; \frac{f_{xx}}{(1+f_x^2)^{3/2}} \qquad \text{for all } x \in [a_1, b),
$$

where

$$
A = \frac{2c_1^2 c_2}{(1+c_1^2)^2}.
$$

Now

$$\lim_{x\to b} \frac{Af_x^2}{(1+f_x^2)^{3/2}} = 0$$

monotonically, therefore there exist an $a_2 \in [a_1, b)$ and a constant $c_4 > 0$ such that $f_x(a_2) > 0$ and $c_4 \leqslant f_{xx}/f_x^3$ for all $x \in [a_2, b)$. The function $\psi = f_x$ satisfies on $[a_2, b)$ all the hypotheses of the previous lemma. Hence $b < \infty$, and $\int_{a_2}^{b} f_x dx$ exists and is finite. It follows that the divergent ray l on F, defined above for $x_0 = a_2$, has length

$$L(l) = \int_{a_2}^{b} (1+f_x^2)^{1/2} dx \leqslant b - a_2 + \int_{a_2}^{b} f_x dx < \infty$$

which contradicts the completeness of F.

Remarks. The Theorem was obtained while investigating the following conjecture by J. Milnor (cf. [1]):

CONJECTURE. *Let F be a complete umbilic-free surface, C^2-immersed in E^3 with K not vanishing identically and either $K \geqslant 0$ throughout or $K \leqslant 0$ throughout. Then*

$$\inf\{(\varkappa_1^2 + \varkappa_2^2)(P) \mid P \in F\} = 0.$$

Our theorem verifies this conjecture for a very special class of surfaces. In fact, this class is so special that none of the hypotheses in the conjecture are needed except, of course, for the completeness. Simple examples show, however, that all the hypotheses are necessary in general. Consult [2] for more details on this as well as recent partial verifications. In this connection it seems to be of interest that our class of translation surfaces contains many complete, strictly convex ($K > 0$), non-compact surfaces without umbilics. We give some examples:

$z(x, y) = e^x + e^y$ over the whole plane,

$z(x, y) = \tan^2 x + y^{-1}$ over $(-\pi/2, \pi/2) \times (0, \infty)$,

$z(x, y) = a^{-1}\ln(\cos ax) + b^{-1}\ln(\cos by)$ over the interval $(-\pi/2a, /\pi 2a) \times \times (-\pi/2b, \pi/2b)$ for $a > 0$, $b > 0$ and $a \neq b$.

By Stoker's structure theorem [3], a complete, non-compact surface F in E^3 with $K > 0$ is the boundary of an unbounded convex body B and it permits a global Cartesian representation $z = z(x, y)$ with $z \geqslant 0$ over a convex domain D of the (x, y)-plane. We note, in passing, that if D contains discs of arbitrarily large radius, then, indeed, the infimum of $\varkappa_2^2 + \varkappa_1^2$ over F is zero. To prove this, take a circle of radius ε^{-1} contained n D, where ε is a given positive number. Consider the sphere with this circle as equator. Move this sphere upwards in the direction perpendicular to the (x, y)-plane until it is entirely within the convex body B. Now lower it until it first touches the surface F at a point P_0. At that point we have obviously $|\varkappa_1| \leqslant \varepsilon$ and $|\varkappa_2| \leqslant \varepsilon$. Since ε was arbitrary, the assertion is proved. Note that we did not have to assume that F has no umbilics.

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TECHNISCHE UNIVERSITÄT BERLIN

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