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SOME SUFFICIENT CONDITIONS FOR UNIVALENCE AND STARLIKENESS

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Let A denote the class of functions $f(z)$ regular in the open unit disc $E = \{z: |z| < 1\}$ and normalized so that $f(0) = 0 = f'(0)-1$. We denote by S the subclass of A consisting of univalent functions in E; C and S^* stand for the subclasses of S whose members are close-to-convex and starlike (with respect to the origin) in E, respectively.

In this note we shall establish a few sufficient conditions for univalence. Some of these conditions are new and others are improvements of the well-known ones.

The basic tool in proving our results is the following lemma due to Jack [1]:

LEMMA. *Let $w(z)$ be regular in the unit disc E and such that $w(0) = 0$. Then if $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point z_0, we have $z_0 w'(z_0) = kw(z_0)$, where $k \geqslant 1$ is a real number.*

THEOREM 1. *If $f \in A$ and*

$$|f'(z)-1|^{1-\gamma}|zf''(z)|^{\gamma} < 1, \quad z \in E, \tag{1}$$

for some $\gamma \geqslant 0$, then f is close-to-convex and bounded in E.

Proof. To prove the assertion it suffices to show that (1) implies

$$|f'(z)-1| < 1, \quad z \in E. \tag{2}$$

Let us define w in E by

$$w(z) = f'(z)-1. \tag{3}$$

Then, clearly, $w(0) = 0$ and $w(z)$ is regular in E. We want to prove that $|w(z)| < 1$ in E. Differentiating (3) we obtain $zf''(z) = zw'(z)$, and therefore

$$|f'(z)-1|^{1-\gamma}|zf''(z)|^{\gamma} = |w(z)|^{1-\gamma}|zw'(z)|^{\gamma}. \tag{4}$$

Suppose that there exists a point z_0 in E such that

$$\max_{|z|\leqslant|z_0|}|w(z)| = |w(z_0)| = 1.$$

Applying Jack's lemma to $w(z)$ at z_0 and letting $z_0w'(z_0)/w(z_0) = k$ so that $k \geqslant 1$, we obtain from (4)

$$|f'(z_0)-1|^{1-\gamma}|z_0f''(z_0)|^{\gamma} = k \geqslant 1, \quad \gamma \geqslant 0,$$

which contradicts (1). Therefore, $|w(z)| < 1$ in E, and so $|f'(z)-1| < 1$, $z \in E$, which shows that f is close-to-convex (and hence univalent) in E. From $|f'(z)-1| < 1$, $z \in E$, it follows easily that f is bounded in E.

Taking $\gamma = 1$ in Theorem 1 we have

COROLLARY 1. *If $f \in A$ and*

$$(5) \qquad |zf''(z)| < 1, \quad z \in E,$$

then f is close-to-convex and bounded in E.

Remark 1. It is readily seen that if $f \in A$ satisfies (5), then f maps the disc $|z| < 1/2$ onto a convex domain. Indeed, from (5) we obtain

$$(6) \qquad zf''(z) = z\varphi(z),$$

where φ is regular and $|\varphi(z)| \leqslant 1$ in E. Integrating (6) we get

$$f'(z)-1 = \int_0^z \varphi(t)\,dt.$$

Therefore,

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{z\varphi(z)}{1+\int_0^z \varphi(t)\,dt}\right| \leqslant \frac{r}{1-r}, \quad r = |z|,$$

from which we deduce that for $r = |z| < 1/2$

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > 0.$$

Consequently, f maps the disc $|z| < 1/2$ onto a convex domain. The function $f(z) = z+z^2/2$, which satisfies (5), shows that the number $1/2$ cannot be replaced by any larger one.

THEOREM 2. *If $f \in A$ satisfies*

$$(7) \qquad |f'(z)-1|^{1-\gamma}\left|1+\frac{zf''(z)}{f'(z)}\right|^{\gamma} < \left(\frac{3}{2}\right)^{\gamma}, \quad z \in E,$$

for some $\gamma \geqslant 0$, then f is close-to-convex and bounded in E.

Proof. It suffices to show that (7) implies (2) which, in turn, proves that f is close-to-convex and bounded in E. To this aim we define w in E by (3) and proceed as in the proof of Theorem 1.

Taking $\gamma = 1$ in Theorem 2, we have

COROLLARY 2. *If $f \in A$ satisfies*

$$\left|1+\frac{zf''(z)}{f'(z)}\right| < \frac{3}{2}, \quad z \in E,$$

then f is close-to-convex and bounded in E.

THEOREM 3. *If $f \in A$ satisfies in E the condition*

$$\left|\frac{zf'(z)}{f(z)}-1\right|^{1-\gamma}\left|\frac{zf''(z)}{f'(z)}\right|^{\gamma} < \left(\frac{3}{2}\right)^{\gamma} \tag{8}$$

for some $\gamma \geqslant 0$, then $f \in S^*$.

Proof. We have to prove that (8) implies the inequality

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > 0, \quad z \in E.$$

Define w in E by

$$G(z) = \frac{zf'(z)}{f(z)} = \frac{1+w(z)}{1-w(z)}. \tag{9}$$

Evidently, $w(0) = 0$. Differentiating (9) logarithmically and simplifying, we obtain

$$\left|\frac{zf'(z)}{f(z)}-1\right|^{1-\gamma}\left|\frac{zf''(z)}{f'(z)}\right|^{\gamma} = \left|\frac{2w(z)}{1-w(z)}\right|\left|1+\frac{zw'(z)}{w(z)}\frac{1}{1+w(z)}\right|^{\gamma}. \tag{10}$$

If $\mathrm{Re}G(z_0) = 0$ for a certain $z_0 \in E$ and $\mathrm{Re}G(z) > 0$ for $|z| < |z_0|$, then $|w(z)| < |w(z_0)| = 1$ for $|z| < |z_0|$ and, of course, $w(z_0) \neq 1$. Applying Jack's lemma to $w(z)$ at the point z_0 and letting $z_0 w'(z_0) = kw(z_0)$ so that $k \geqslant 1$, we obtain from (10)

$$\left|\frac{z_0 f'(z_0)}{f(z_0)}-1\right|^{1-\gamma}\left|\frac{z_0 f''(z_0)}{f(z_0)}\right|^{\gamma} \geqslant \left(1+\frac{k}{2}\right)^{\gamma} \geqslant \left(1+\frac{1}{2}\right)^{\gamma} \geqslant \left(\frac{3}{2}\right)^{\gamma}, \quad \gamma \geqslant 0,$$

which contradicts (8). This proves that $\mathrm{Re}G(z) > 0$ in E, and hence $f \in S^*$. Thus the proof of Theorem 3 is completed.

Taking $\gamma = 1$ in Theorem 3, we have

COROLLARY 3. *If $f \in A$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{3}{2}, \quad z \in E,$$

then f is starlike univalent in E.

THEOREM 4. *If $f \in A$ satisfies*

$$\left|a\left(\frac{zf'(z)}{f(z)} - 1\right) + (1-a)\frac{z^2 f''(z)}{f(z)}\right| < 1, \quad z \in E, \tag{11}$$

for $0 \leqslant a \leqslant 1$, then f is bounded and starlike in E.

Proof. It is sufficient to prove that (11) implies the inequality

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \quad z \in E.$$

We define w in E by

$$w(z) = \frac{zf'(z)}{f(z)} - 1. \tag{12}$$

Evidently, $w(0) = 0$. Differentiating (12) logarithmically we obtain

$$\frac{zf''(z)}{f'(z)} = w(z) + \frac{zw'(z)}{1+w(z)},$$

and hence

$$\left|a\left(\frac{zf'(z)}{f(z)} - 1\right) + (1-a)\frac{z^2 f''(z)}{f(z)}\right| = |w(z)|\left|1 + (1-a)\left(w(z) + \frac{zw'(z)}{w(z)}\right)\right|. \tag{13}$$

We claim that $|w(z)| < 1$, $z \in E$. Suppose z_0 is a point of E such that

$$\max_{|z| \leqslant |z_0|} |w(z)| = |w(z_0)| = 1.$$

Applying Jack's lemma to $w(z)$ at the point z_0, letting $z_0 w'(z_0)/w(z_0) = k$ so that $k \geqslant 1$ and $w(z_0) = e^{i\theta}$, we obtain from (13)

$$\left|a\left(\frac{z_0 f'(z_0)}{f(z_0)} - 1\right) + (1-a)\frac{z_0^2 f''(z_0)}{f(z_0)}\right| = |1 + (1-a)(k + e^{i\theta})| \geqslant 1,$$

which contradicts (11). This proves that $|w(z)| < 1$ in E, and hence $|zf'(z)/f(z) - 1| < 1$ in E, which implies that f is bounded and starlike in E.

Taking $\alpha = 0$ in Theorem 4, we get

Corollary 4. *If $f \in A$ satisfies*

$$\left|\frac{z^2 f''(z)}{f(z)}\right| < 1, \quad z \in E, \tag{14}$$

then $f \in S^*$.

Theorem 5. *If $f \in A$, $\alpha > 1/2$, and*

$$\operatorname{Re}\left[\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\frac{1}{f'(z)}\right] < \frac{1+2\alpha}{2}, \quad z \in E, \tag{15}$$

then f is close-to-convex and bounded in E.

Proof. It suffices to prove that (15) implies (2) from which our result follows. To this aim we define w in E by (3) and proceed as in the proof of Theorem 1.

Taking $\alpha = 1$ in Theorem 5 we have

Corollary 5. *If $f \in A$ satisfies*

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) < \frac{3}{2}, \quad z \in E, \tag{16}$$

then f is close-to-convex and bounded in E.

Remark 2. Ozaki [2] has proved that if (16) holds, then f is univalent in E. The result of Corollary 5 shows that f is not only univalent but also close-to-convex and bounded in E.

Our next theorem strengthens the result of Corollary 5.

Theorem 6. *If $f \in A$ satisfies* (16), *then*

$$\frac{zf'(z)}{f(z)} \prec \frac{2(1-z)}{2-z}, \quad z \in E. \tag{17}$$

Moreover, f is starlike in E.

Proof. One can easily verify that $g(z) = 2(1-z)/(2-z)$ is univalent in E. Now, let

$$\frac{zf'(z)}{f(z)} = \frac{2\big(1-w(z)\big)}{2-w(z)}, \quad z \in E. \tag{18}$$

Evidently, $w(0) = 0$. We want to prove that $|w(z)| < 1$ in E. Differentiating (18) logarithmically we get

$$1+\frac{zf''(z)}{f'(z)} = \frac{2\big(1-w(z)\big)}{2-w(z)}+\frac{zw'(z)}{2-w(z)}-\frac{zw'(z)}{1-w(z)}. \tag{19}$$

Suppose that z_0 is a point of E such that

$$\max_{|z|\leqslant|z_0|}|w(z)| = |w(z_0)| = 1.$$

Then applying Jack's lemma to $w(z)$ at z_0, letting $z_0 w'(z_0)/w(z_0) = k$ so that $k \geqslant 1$, and $w(z_0) = e^{i\theta}$, we obtain from (19)

$$\operatorname{Re}\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right) = \frac{3}{2}+\frac{3(k-1)}{5-4\cos\theta} \geqslant \frac{3}{2},$$

which contradicts (16). This proves that $|w(z)| < 1$ in E, and hence (17) holds, which in turn implies

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geqslant \min_{|z|=r}\operatorname{Re}\frac{2(1-z)}{2-z} = \frac{2(1-r)}{2-r} > 0, \qquad |z| = r < 1.$$

Hence $f \in S^*$. This completes the proof of Theorem 6.

REFERENCES

[1] I. S. Jack, *Functions starlike and convex of order a*, The Journal of the London Mathematical Society 3 (1971), p. 469-474.

[2] S. Ozaki, *On the theory of multivalent functions. II*, Science Reports of the Tokyo Bunrika Daigaku, Section A, 4 (1941), p. 45-87.

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