COLLOQUIUM MATHEMATICUM

VOL. XXVII 1973 FASC. 1

SOME CURVES OF PRESCRIBED RIM-TYPES

BY

B. B. EPPS, Jr. (HOUSTON, TEXAS)

The theory of continua, when studied from the set-theoretical standpoint, offers a number of concepts in which some cardinality ideas play an important role (see [2], Section 51). Among them probably most natural is the concept of a *rational curve* understood to mean a non--degenerate connected compact metric space having a basis of open sets with countable boundaries. Given a rational curve X, by the *rim-type* of X we understand the minimum ordinal α such that X has a basis of open sets with countable boundaries whose α-th derivatives are empty (cf. [5], p. 296). The rim-types of rational curves are countable ordinals, and the arc is an example of a curve of rim-type 1. It is known that each acyclic rational curve of a finite rim-type contains an arc (see [4], Theorem 1.2). Answering a question asked recently by Lelek (see [3], Problem 13), we show the existence of chainable, thus acyclic, rational curves of any finite rim-type with the property that each subcurve which is not an arc has rim-type equal to that of the curve. The rim-type of a rational curve X will be denoted by $\varrho(X)$, and we shall denote by $C(X)$ the collection of all subcurves of X.

EXAMPLE 1. *For each integer $n > 1$, there exists a chainable rational curve X_n such that*

$$\{\varrho(Y):\ Y \in C(X_n)\} = \{1, n\}.$$

Proof. Let T be the Cantor ternary set of real numbers of the unit interval, and let T^* be the subset of T consisting of all right end points of bounded intervals adjacent to T. In the other words, we have $t \in T^*$ if and only if

$$t = \sum_{i=1}^{k} \frac{t_i}{3^i},$$

where $t_i = 0, 2$ for $i < k$ and $t_k = 2$. For each such a number t, we define three numbers t', t'' and t''' in the following way.

We put

$$t' = \sum_{i=1}^{k} \frac{t_i'}{3^i},$$

where $t_i' = t_i$ for $i < k$ and $t_k' = 1$, and

$$t'' = \left(\sum_{i=1}^{k} \frac{t_i}{2}\right)^{-1}, \quad t''' = \left(n + \sum_{i=1}^{k} \frac{t_i}{2}\right)^{-1}.$$

First, we construct a continuum X on the plane, and then we obtain X_n as the image of X under a monotone mapping. We take the sets

$$\mathscr{C} = \{(t, 0): t \in T\}, \quad \mathscr{J} = \{(0, y): 0 \leqslant y \leqslant 1\}$$

and, for each number $t \in T^*$, we define five points

$$a_t = (t, 0), \quad b_t = (t, t''), \quad c_t = (t', t''), \quad d_t = (t', 0), \quad p_t = (t, t''').$$

Given a pair of points p, q of the plane, we denote by $\overline{pq}$ the straight segment joining p and q. It is not difficult to see that the set

$$X = \mathscr{C} \cup \mathscr{J} \cup \bigcup_{t \in T^*} (\overline{a_t b_t} \cup \overline{b_t c_t} \cup \overline{c_t d_t})$$

is a chainable continuum, irreducible between the point $(1, 0)$ and any point of the segment $\mathscr{J}$. The rim-type of X is infinite. Observe, however, that each segment $\overline{a_t p_t}$ $(t \in T^*)$ can be represented as the intersection of a decreasing sequence of open subsets of X with countable boundaries whose n-th derivatives are empty.

Indeed, given a number $t \in T^*$, open subsets $G_j(t)$ of X having these properties can be defined by the formula

$$G_j(t) = \{(x, y) \in X: t - \varepsilon_j < x < t + \varepsilon_j, \ -\varepsilon_j < y < t''' + \varepsilon_j\},$$

where $\varepsilon_j = 2^{-j}(t - t') = 2^{-j}3^{-k}$ for $j = 1, 2, \ldots$

Let D be the upper semi-continuous decomposition of X into continua such that the only non-degenerate elements of D are $\mathscr{J}$ and $\overline{a_t p_t}$, where $t \in T^*$. We define X_n to be the decomposition space $X_n = X/D$. Thus X_n, as a monotone continuous image of X, is also a chainable continuum (see [1], p. 47), irreducible between exactly one pair of points corresponding to $(1, 0)$ and $(0, 0)$, respectively. Moreover, the above-mentioned observation implies that the rim-type of X_n is now finite and equal to n.

Consider a curve $Y \in C(X_n)$ and assume Y is not an arc. Then, taking the inverse image Y' of Y in X, we can see Y' is not contained in any of the arcs $\mathscr{J}$ and $\overline{a_t b_t} \cup \overline{b_t c_t} \cup \overline{c_t d_t}$ $(t \in T^*)$. But a continuum contained in X and not contained in any of these arcs has to join a pair of points of C which are not end points of intervals adjacent to C. The portion of X lying between such a pair of points is very much like the curve X itself and, by the irreducibility of X, it is contained in each continuum joining these points in X. Consequently, Y' contains a topological copy of X. It follows that Y contains a topological copy of X_n, whence $\varrho(Y) = n$ and the discussion of Example 1 is completed.

Now, denoting by ω the least infinite ordinal, let us introduce the order topology in the set of all ordinals $a \leqslant \omega$. This is the space $\{1, 2, \ldots\} \cup \cup \{\omega\}$ being the one-point compactification of the discrete space $\{1, 2, \ldots\}$.

EXAMPLE 2. *For each closed subset N of the space $\{1, 2, \ldots\} \cup \{\omega\}$ with $1 \in N$, there exists a chainable rational curve X_N such that*

$$\{\varrho(Y)\colon\ Y \in \mathscr{C}(X_N)\} = N.$$

Proof. We distinguish three cases, though case (ii) includes case (i)

(i) If $N = \{1, \omega\}$, then let X_N be the curve X of infinite rim-type as defined in Example 1. Each subcurve of X either is an arc or has rim-type ω. In addition, for the purpose of case (ii), let us also denote the same curve X by X_ω and let us select the points (1, 0) and (0, 0) between which X_ω is irreducible.

(ii) If N is finite, then we have $N = \{n_1, \ldots, n_k\}$, where n_i are positive integers or ω. Let X_1 be an arc, let X_n be the curve constructed in Example 1 for any integer $n > 1$, and let X_ω be the curve from case (i). Each of these curves is irreducible between a selected pair of points. We define X_N to be the union of topological copies of the curves $X_{n_1}, \ldots, X_{n_k}$ joined together like a chain at points corresponding to the above-selected points of irreducibility.

(iii) If N is infinite, then we have $N = \{n_1, n_2, \ldots\} \cup \{\omega\}$, where $1 = n_1 < n_2 < \ldots$ are integers. Similarly to what was done in case (ii), we define X_N to be the union of an infinite chain of topological copies of the curves $X_{n_1}, X_{n_2}, \ldots$ plus a point q such that q does not belong to any of these copies and they are required to converge to q.

REFERENCES

[1] R. H. Bing, *Concerning hereditarily indecomposable continua,* Pacific Journal of Mathematics 1 (1951), p. 43-51.

[2] K. Kuratowski, *Topology,* vol. II, New York 1968.

[3] A. Lelek, *Some problems concerning curves,* Colloquium Mathematicum 23 (1971), p. 93-98.

[4] — and L. Mohler, *On the topology of curves III,* Fundamenta Mathematicae 71 (1971), p. 147-160.

[5] K. Menger, *Kurventheorie,* Bronx 1967.

UNIVERSITY OF HOUSTON

Reçu par la Rédaction le 26. 8. 1971