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THE BOURBAKI INTEGRAL AS MAXIMAL INTEGRAL

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1. Introduction. A theory of maximal embedded absolutely convergent integral was developed in [5]-[8] and [10]. The present paper, which is a continuation of [6], contains also chapter 2 of the unpublished paper [8] and gives the proof of theorem 4 of [11].

Let the pair $(\mathfrak{M}, \mathfrak{W})$ consist of a σ-vector-lattice $\mathfrak{M}$ and a conditionally complete half-ordered vector space $\mathfrak{W}$, and let $\mathfrak{B}$ be a subvector-lattice of $\mathfrak{M}$.

$\mathfrak{B}$ is called *embedded* (= eingelagert [6]) in $\mathfrak{M}$ iff for every $0 \leqslant f \in \mathfrak{M} - \mathfrak{B}$ there exists a sequence $0 \leqslant v_1 \leqslant v_2 \leqslant \ldots \leqslant f$, $v_i \in \mathfrak{B}$ $(i = 1, 2, \ldots)$, such that $\sup\limits_i v_i \in \mathfrak{M} - \mathfrak{B}$.

A linear, positive, continuous mapping $I\colon \mathfrak{B} \to \mathfrak{W}$ is called an *absolutely convergent integral $I/\mathfrak{B}$ embedded in the pair* $(\mathfrak{M}, \mathfrak{W})$. Such an integral is said to be *maximal* iff there exists for any vector-space $\mathfrak{R}$ with the property $\mathfrak{B} \subseteq \mathfrak{R} \subseteq \mathfrak{M}$ either no linear, positive and continuous extension of $I\colon \mathfrak{B} \to \mathfrak{W}$ to $\mathfrak{R}$ or more than one such an extension. *Continuity* of I means that the implication $\mathfrak{B} \ni v_i \downarrow 0 \Rightarrow I(v_i) \downarrow 0$ is valid.

Now let Ω be a locally compact space, $\mathfrak{E}$ the vector-lattice of real continuous functions $e\colon \Omega \to R$ with compact support. R is the set of real numbers (without $\pm\infty$). The *support* of a function e is the closure of the set $\{x; x \in \Omega \text{ and } e(x) \neq 0\}$. A linear and positive functional $E\colon \mathfrak{E} \to R$ is called a *Bourbaki elementary integral $E/\mathfrak{E}$* or a *positive Radon measure.* For every non-negative and lower-continuous function $s\colon \Omega \to R \cup \cup \{+\infty, -\infty\}$ there is defined the number

$$E(s)\colon = \sup\{E(e);\ 0 \leqslant e \leqslant s,\ e \in \mathfrak{E}\}.$$

For any numerical function $f\colon \Omega \to R \cup \{+\infty, -\infty\}$,

$$N_E(f)\colon = \inf\{E(s);\ |f| \leqslant s \text{ and } s \text{ lower-continuous on } \Omega\}$$

is called the *semi-norm* of f. Let $\mathfrak{F}_E$ be the set of numerical functions $f\colon \Omega \to R \cup \{+\infty, -\infty\}$ having a finite semi-norm $N_E(f)$. The closure $\tilde{\mathfrak{S}}_E$ of $\mathfrak{E}$ in $\mathfrak{F}_E$ with regard to N_E is called the *set of integrable numerical func-*

tions on Ω relative to $E/\mathfrak{E}$. The integral $I_E(f)$, where $f \in \tilde{\mathfrak{S}}_E$, is obtained by continuous extension of the mapping E. Let $\mathfrak{S}_E$ denote the set of integrable real functions $f\colon \Omega \to R$. The pair $I_E/\mathfrak{S}_E$ is called a *Bourbaki integral relative to* $E/\mathfrak{E}$. Finally, let $\mathfrak{M}_E$ denote the σ-vector-lattice of measurable real functions on Ω relative to $E/\mathfrak{E}$, $\mathfrak{A}_E$ the σ-field of measurable subsets of Ω relative to $E/\mathfrak{E}$, and m_E the measure belonging to $E/\mathfrak{E}$.

The main problem of our paper is to characterize those elementary Bourbaki integrals $E/\mathfrak{E}$ for which the Bourbaki integral $I_E/\mathfrak{S}_E$ is a maximal absolutely convergent integral embedded in the pair $(\mathfrak{M}_E, R)$. With such a characterisation in hand we are in a position to decide already at the level $E/\mathfrak{E}$ of the extension-procedure $E/\mathfrak{E} \to I_E/\mathfrak{S}_E$ whether the result of this procedure will be maximal or not.

The problem is solved by theorems 1-4 asserting the equivalence of conditions III-VIII. Conditions I-IV are auxiliary ones and serve to get the equivalence of condition V and the maximality of $I_E/\mathfrak{S}_E$ (condition VI). An essential part (conditions II and IV) lies in the use of the notion of a local m_E-nullset. We characterize, by means of $E/\mathfrak{E}$, those measures m_E for which every local m_E-nullset is an m_E-nullset (lemma 3). In the definition of local m_E-nullsets the compact subsets of Ω play a great role. In order to use only objects given directly by $E/\mathfrak{E}$ we introduce the set $\mathfrak{L}_{\mathfrak{E}}$ (definition 1). Then we get condition VII equivalent to the maximality of $I_E/\mathfrak{S}_E$. Thereby the constructively defined Bourbaki integral is arranged in a sufficiently general manner in the framework of maximal embedded absolutely convergent integral.

2. Characterisation of the Bourbaki integral $I_E/\mathfrak{S}_E$ as a maximal embedded absolutely convergent integral. A set $A \subseteq \Omega$ is known to be a local m_E*-nullset* if for every point $x \in \Omega$ there exists a neighbourhood $U(x)$ of x such that $U(x) \cap A$ is of m_E-measure zero [1].

LEMMA 1. *The following conditions are equivalent*:

I. *Let* $A \in \mathfrak{A}_E$ *and* $m_E(A) = +\infty$.
Then $\mathfrak{A}_E \ni B \subseteq A$ *implies* $m_E(B) = 0$ *or* $m_E(B) = +\infty$.

II. A *is a local* m_E*-nullset but not a* m_E*-nullset.*

Proof. I $\Rightarrow$ II. Let $x \in \Omega$. Then there exists a compact neighbourhood $U(x)$ of x. Since $m_E(U(x)) < +\infty$, there is $m_E(A \cap U(x)) < +\infty$ and condition I implies $m_E(A \cap U(x)) = 0$. So A is a local m_E-nullset. Because of $m_E(A) = +\infty$, A is not of m_E-measure zero.

II $\Rightarrow$ I. Assume II. By virtue of [1] (cf. remark after definition 2 and Corollaire 1, p. 183), we have $A \in \mathfrak{A}_E$ and $m_E(A) = +\infty$. Let $\mathfrak{A}_E \ni B \subseteq A$. Since $m_E(B) = +\infty$ implies I immediately, consider the case $m_E(B) < +\infty$. We have to show that $m_E(B) = 0$. Since B is a subset of a local m_E-nullset, B also is a local m_E-nullset. Therefore, by [1] (cf. Corollaire 1, p. 183), B is an m_E-nullset.

LEMMA 2. *The following conditions are equivalent:*

III. *The measure m_E is weakly σ-finite, i.e. for every m_E-measurable set B with $m_E(B) = +\infty$ there exists a sequence $B_1 \subseteq B_2 \subseteq \ldots \subseteq B$ of m_E-measurable sets B_i of finite measure such that $m_E(\bigcup_i B_i) = +\infty$.*

IV. *Every local m_E-nullset is an m_E-nullset.*

Proof. Use Lemma 1 and Theorem 1 of [9]. See also [12].

LEMMA 3. *Condition* IV *is equivalent to the condition:*

V. *Let $A \subseteq \Omega$. If for every compact set $K \subseteq \Omega$ and every number $\varepsilon > 0$ there exists a sequence $0 \leqslant e_1^{(K)} \leqslant e_2^{(K)} \leqslant \ldots, e_i^{(K)} \in \mathfrak{E}$, $(i = 1, 2, \ldots)$, such that $\chi_{A \cap K} \leqslant \sup_i e_i^{(K)}$ and $\sup_i E(e_i^{(K)}) < \varepsilon$, then there exists a sequence $0 \leqslant e_1 \leqslant e_2 \leqslant \ldots, e_i \in \mathfrak{E}$, $(i = 1, 2, \ldots)$, such that $\chi_A \leqslant \sup_i e_i$, where χ_A is the characteristic function of the set A.*

Proof. V $\Rightarrow$ IV. Let $A \subseteq \Omega$ be a local m_E-nullset. Then, because of [1] (proposition 5, p. 183), $m_E^*(A \cap K) = 0$ for every compact subset K of Ω. Since m_E^* is the outer measure belonging to m_E, the first part of condition V is evidently valid. Hence there exists a sequence $0 \leqslant e_1 \leqslant e_2 \leqslant \ldots, e_i \in \mathfrak{E}$ $(i = 1, 2, \ldots)$ such that $\chi_A \leqslant \sup_i e_i$. As a local m_A-nullset, A is m_E-measurable. Therefore also the functions $s_i := \chi_A \cdot e_i$ are m_E-measurable and we have $0 \leqslant s_1 \leqslant s_2 \leqslant \ldots \uparrow \chi_A$.

Now define the sets

$$S_i := \{x; x \in \Omega \text{ and } s_i(x) > 0\} \quad i = 1, 2, \ldots$$

We have $m_E(S_i) < +\infty$. Because of $S_1 \subseteq S_2 \subseteq \ldots \subseteq A$ and $s_i \uparrow \chi_A$, we get $\bigcup_i S_i = A$. Hence

$$\sup_i m_E(S_i) = m_E(A).$$

In the case of $m_E(A) < +\infty$ we conclude, using [1] (Corollaire 1, p. 183), that A is an m_E-nullset. Now let $m_E(A) = +\infty$. Then there exists an integer n such that $0 < m_E(S_n) < +\infty$, where S_n is an open set. We have $\chi_{S_n} \in \mathfrak{S}_E$ and χ_{S_n} is lower-continuous. Therefore

$$0 < m_E(S_n) = \sup\{E(e); 0 \leqslant e \leqslant \chi_{S_n}, e \in \mathfrak{E}\}.$$

In any case there exists a function $e \in \mathfrak{E}$ with the properties $0 \leqslant e \leqslant \chi_{S_n}$ and $E(e) > 0$. Consider the sets $A_k := \{x; e(x) \geqslant 1/k\}$, $k = 1, 2, \ldots$ There is $A_k \subseteq S_n \subseteq T(e_n)$, where $T(e_n)$ is the support of the function e_n. A_k is closed, therefore compact. Evidently, there is an integer k such that $m_E(A_k) > 0$. Since A is a local m_E-nullset by hypothesis, we have a contradiction (cf. [1], proposition 5, p. 183).

IV $\Rightarrow$ V. If the first part of condition V holds for the set $A \subseteq \Omega$, then, by virtue of [1] (proposition 5, p. 183), we infer that A is a local m_E-nullset. Therefore, by IV, $m_E(A) = 0$. Consequently, there exists a sequence $0 \leqslant e_1 \leqslant e_2 \leqslant \ldots, e_i \in \mathfrak{C}$ $(i = 1, 2, \ldots,)$ with the property $\chi_A \leqslant \sup\limits_i e_i$.

THEOREM 1. (*First characterization of the Bourbaki integral as a maximal integral.*)

Let $E/\mathfrak{C}$ be a Bourbaki elementary integral on the locally compact space Ω. Then condition V *is equivalent to the condition*

VI. *The Bourbaki integral $I_E/\mathfrak{S}_E$ belonging to $E/\mathfrak{C}$ is a maximal absolutely convergent integral embedded in $(\mathfrak{M}_E, R)$.*

Proof. V $\Rightarrow$ VI. Assuming condition V, we have, in view of lemma 3, condition IV and, by lemma 2, condition III. Now condition VI follows by virtue of theorem 6 of [9] and the fact that $I_E/\mathfrak{S}_E \equiv I_{m_E}/\mathfrak{S}_{m_E}$, where $I_{m_E}/\mathfrak{S}_{m_E}$ is the integral belonging to the measure m_E in the sense of [4].

VI $\rightarrow$ V. Assuming condition VI we infer, by condition III, and, by virtue of lemma 2, condition IV, whence condition V follows by lemma 3.

The following Corollary shows that condition V is satisfied in some important cases.

COROLLARY. *Condition* V *holds in the following cases*:

1. *Ω is compact.*

2. *Ω is countable at infinity, i.e. there exists a sequence (K_i) $(i = 1, 2, \ldots)$ of compact subsets K_i such that $\Omega = \bigcup\limits_i K_i$.*

Proof. Ad 1. If Ω is compact, then $\chi_\Omega \equiv 1 \in \mathfrak{C}$. Therefore $\chi_A \leqslant \sup\limits_i e_i$, where $e_i = 1$ $(i = 1, 2, \ldots)$ for every subset A of Ω.

Ad 2. If Ω is countable at infinity, then there exists a sequence (U_i) of relatively compact sets U_i $(i = 1, 2, \ldots)$ such that $\Omega = \bigcup\limits_i U_i$, $U_1 \subseteq U_2 \subseteq \ldots$, $U_i \subseteq U_{i+1}$ $(i = 1, 2, \ldots)$. Since Ω is locally compact, we can find a continuous function e_1 defined on Ω with $e_1 \geqslant 0$, $e_1(x) = 1$ for $x \in \overline{U}_1$, and $e_1(x) = 0$ for $x \notin U_2$. The compactness of $\overline{U}_1$ implies $e_1 \in \mathfrak{C}$. In the same manner we can find a continuous function e_2 defined on Ω with the properties $e_2 \geqslant 0$, $e_2(x) = 1$ for $x \in \overline{U}_2$, and $e_2(x) = 0$ for $x \notin U_3$. Again $e_2 \in \mathfrak{C}$. Proceeding in this way we get a sequence (e_i) of functions $e_i \in \mathfrak{C}$ such that $e_1 \leqslant e_2 \leqslant \ldots$ and, evidently, $\sup\limits_i e_i \geqslant \chi_A$ for every subset $A \subseteq \Omega$. Condition V holds.

Remark. In the case of our corollary the only reason for maximality of the Bourbaki integral $I_E/\mathfrak{S}_E$ is the space Ω with its topology and not the functional E.

Now we want to have a condition equivalent to the maximality of $I_E/\mathfrak{S}_E$ which uses sets given directly by $\mathfrak{C}$ only.

Definition 1. Let $\mathfrak{C}$ be the vector-lattice of continuous functions with compact support on the locally compact space Ω. We define the set $\mathfrak{L}_{\mathfrak{C}}$ as follows:

A subset A of Ω *belongs to* $\mathfrak{L}_E$ iff there is a function $e \in \mathfrak{E}$, $e \geqslant 0$, and there are real numbers $c_1 > c_2 > 0$ such that

$$A = \{x;\, x \in \Omega \text{ and } c_1 \geqslant e(x) \geqslant c_2\}.$$

THEOREM 2. *Condition* V *is equivalent to the condition*

VII. *Let* $A \subseteq \Omega$. *If there exists a sequence* $0 \leqslant e_1^{(L)} \leqslant e_2^{(L)} \leqslant \ldots,\, e_i^{(L)} \in \mathfrak{E}$ $(i = 1, 2, \ldots)$ *for every set* $L \in \mathfrak{L}_{\mathfrak{E}}$ *and every number* $\varepsilon > 0$ *such that*

$$\chi_{A \cap L} \leqslant \sup_i e_i^{(L)} \quad \textit{and} \quad \sup_i E(e_i^{(L)}) < \varepsilon,$$

then there exists a sequence $0 \leqslant e_1 \leqslant e_2 \leqslant \ldots,\, e_i \in \mathfrak{E}$ $(i = 1, 2, \ldots)$ *such that*

$$\chi_A \leqslant \sup_i e_i.$$

Proof. V $\Rightarrow$ VII. Let $A \subseteq \Omega$ be arbitrary and let the first part of condition VII hold. We have to show that the first part of condition V holds. Let $K \subseteq \Omega$ be a compact set and $\varepsilon > 0$. If $K = \Omega$, then $\chi_\Omega \in \mathfrak{E}$ and $\chi_\Omega \geqslant \chi_A$. So let $\Omega - K \neq \emptyset$. There exists a compact neighbourhood U_k of K, $K \subseteq U_K$; for the interior U_K^0 of U_K we have $K \subseteq U_K^0$. Now there exists a continuous function e on Ω having the following properties: $e(x) \geqslant 0$ for all $x \in \Omega$, $e(x) = 1$ for all $x \in K$, and $e(x) = 0$ for all $x \notin U_K^0$. Hence $e \in \mathfrak{E}$. Putting

$$L_i := \{x;\, i \geqslant e(x) \geqslant 1/i\}, \quad i = 1, 2, \ldots,$$

we have

$$K \subseteq \bigcup_i L_i.$$

Since the first part of VII holds for the set A, for every set L_i and every number $\varepsilon/2^i$ $(\varepsilon > 0)$ there exists a sequence $0 \leqslant e_1^{(L_i)} \leqslant e_2^{(L_i)} \leqslant \ldots,$ $e_i^{(L_i)} \in \mathfrak{E}$, such that

$$\chi_{A \cap L_i} \leqslant \sup_\varrho e_i^{(L_i)} \quad \text{and} \quad \sup_\varrho E(e_i^{(L_i)}) < \varepsilon/2^i.$$

Defining the functions $e_\varrho^{(K)}$ by the equality

$$e_\varrho^{(K)} := \sum_{i=1}^{\varrho} e_\varrho^{(L_i)},$$

we have $e_\varrho^{(K)} \in \mathfrak{E}$ for all ϱ and $0 \leqslant e_1^{(K)} \leqslant e_2^{(K)} \leqslant \ldots$ Since $K \subseteq \bigcup_i L_i$,

$$\sup_\varrho e_\varrho^{(K)} \geqslant \chi_{A \cap K}.$$

Moreover,

$$E(e_\varrho^{(K)}) = \sum_{i=1}^{\varrho} E(e_\varrho^{(L_i)}) < \sum_{i=1}^{\varrho} \varepsilon/2^i$$

and

$$\sup_{\varrho} E(e_{\varrho}^{(K)}) < \varepsilon,$$

whence the first part of condition V follows. On account of V there exists a sequence $0 \leqslant e_1 \leqslant e_2 \leqslant \ldots$, $e_i \in \mathfrak{E}$, such that

$$\sup_i e_i \geqslant \chi_A.$$

VII $\Rightarrow$ V. Let $A \subseteq \Omega$ and assume that the first part of condition V is valid. We have to show the first part of condition VII. Let $L \in \mathfrak{L}_{\mathfrak{E}}$ and let $\varepsilon > 0$ be arbitrary. Since the support of the function e is compact, the set L is compact. Then there exists a sequence $e_1^{(L)} \leqslant e_2^{(L)} \leqslant \ldots$, $0 \leqslant e_i^{(L)} \in \mathfrak{E}$ $(i = 1, 2, \ldots)$ such that

$$\sup_i e_i^{(L)} \geqslant \chi_{A \cap L} \quad \text{and} \quad \sup_i E(e_i^{(L)}) < \varepsilon.$$

Condition VII implies now the existence of a sequence $0 \leqslant e_1 \leqslant e_2 \leqslant \ldots$, $e_i \in \mathfrak{E}$ $(i = 1, 2, \ldots)$ with the required property.

THEOREM 3 (Second characterisation of the Bourbaki integral as a maximal integral). *Conditions* VI *and* VII *are equivalent.*

Proof is a consequence of theorems 1 and 2.

THEOREM 4 (Third characterisation of the Bourbaki integral as a maximal integral). *Condition* VI *is equivalent to the condition*

VIII. *If $I/\mathfrak{V}$ is a maximal absolutely convergent integral embedded in the pair $(\mathfrak{M}_E, R)$ and having the property $E/\mathfrak{E} \subseteq I/\mathfrak{V}$, i.e. $\mathfrak{E} \subseteq \mathfrak{V}$ and $E(e) = I(e)$ for every $e \in \mathfrak{E}$, then $I/\mathfrak{V} \equiv I_E/\mathfrak{S}_E$.*

Proof. VI $\Rightarrow$ VIII. Let $I/\mathfrak{V}$ be a maximal absolutely convergent integral embedded in $(\mathfrak{M}_E, R)$ and such that $E/\mathfrak{E} \subseteq I/\mathfrak{V}$. It follows from the main criterion on maximal embedded absolutely convergent integrals [6] that the theorem of B. Levi is true for $I/\mathfrak{V}$. In particular, since $E/\mathfrak{E} \subseteq I/\mathfrak{V}$, we have the implication

$$0 \leqslant e_1 \leqslant e_2 \leqslant \ldots \uparrow f \in \mathfrak{M}_E, \qquad e_i \in \mathfrak{E} \ (i = 1, 2, \ldots),$$

and the existence of $\sup_i E(e_i) < +\infty$ imply

$$f \in \mathfrak{V} \quad \text{and} \quad \sup_i E(e_i) = I(f).$$

We know that for $f \in \mathfrak{M}_E$ there is $f \in \mathfrak{S}_E$ iff $N_E(f) < +\infty$. Let $0 \leqslant f \in \mathfrak{S}_E$. Hence we have $N_E(f) < +\infty$ and therefore there exists a sequence $0 \leqslant e_1 \leqslant e_2 \leqslant \ldots$, $e_i \in \mathfrak{E}$ $(i = 1, 2, \ldots)$, such that

$$\sup_i e_i \geqslant f \quad \text{and} \quad \sup_i E(e_i) = I(\sup_i e_i) \geqslant N_E(f) \geqslant 0.$$

We have

$$\sup_i e_i \in \mathfrak{V}$$

and because of [6], theorem 3, $\mathfrak{V}$ is an ideal in $\mathfrak{M}_E$. Hence $f \in \mathfrak{V}$. Hence and from the properties of a vector-lattice we infer $\mathfrak{S}_E \subseteq \mathfrak{V}$. By the definition,

$$N_E(f) := \inf\{\sup E(e_i);\ 0 \leqslant e_1 \leqslant e_2 \leqslant \ldots,\ e_i \in \mathfrak{E},\ \sup_i e_i \geqslant f\}.$$

Hence and from the theorem of B. Levi we easily conclude that $I(f) \leqslant I_E(f)$ for every $0 \leqslant f \in \mathfrak{S}_E$. On the other hand, however, it is impossible that $I(f) < I_E(f)$ for a certain $0 \leqslant f \in \mathfrak{S}_E$. In fact, for if there is $0 \leqslant f \in \mathfrak{S}_E$ with $I(f) < I_E(f)$ we define $F(g) := I_E(g) - I(g)$, $g \in \mathfrak{S}_E$. Then F is linear, positive and continuous on $\mathfrak{S}_E$ and F is not identically zero. Moreover, we have $F(e) = 0$ for every $e \in \mathfrak{E}$. But for every $0 \leqslant g \in \mathfrak{S}_E$ there exists a sequence $0 \leqslant e_1 \leqslant e_2 \leqslant \ldots$, $e_i \in \mathfrak{E}$ $(i = 1, 2, \ldots)$, such that $g \leqslant \sup_i e_i$. Hence $0 \leqslant F(g) \leqslant \sup_i F(e_i) = 0$ for every $g \in \mathfrak{S}_E$. Therefore the inequality $I(f) < I_E(f)$ for a $0 \leqslant f \in \mathfrak{S}_E$ is impossible.

Till now we have proved the following facts: $\mathfrak{S}_E \subseteq \mathfrak{V}$ and $I_E(s) = I(s)$ for every $s \in \mathfrak{S}_E$. Assuming $\mathfrak{V} - \mathfrak{S}_E \neq \emptyset$ we can find a $v \in \mathfrak{V}$ with $v \notin \mathfrak{S}_E$. Then we have $v^+ \notin \mathfrak{S}_E$ or $v^- \notin \mathfrak{S}_E$, where $v^+ := \sup(v, 0)$ and $v^- := \sup(-v, 0)$. Without loss of generality we may assume that $0 \leqslant v \in \mathfrak{V} - \mathfrak{S}_E$. Since $\mathfrak{S}_E$ is embedded in $\mathfrak{M}_E$, there exists a sequence $0 \leqslant s_1 \leqslant s_2 \leqslant \ldots \leqslant v$, $s_i \in \mathfrak{S}_E$, such that $\sup_i s_i \in \mathfrak{M}_E - \mathfrak{S}_E$. By virtue of theorem 3 of [6], $\mathfrak{V}$ is an ideal and, therefore, $\sup_i s_i \in \mathfrak{V}$. But the theorem of B. Levi implies

$$\sup_i I_E(s_i) = +\infty.$$

Hence

$$\sup_i I(s_i) = +\infty.$$

On the other hand, we infer from $s_i \leqslant v (i = 1, 2, \ldots)$ and $v \in \mathfrak{V}$ that $I(s_i) \leqslant I(v) < +\infty$ for every i, whence

$$\sup_i I(s_i) < +\infty.$$

But this is a contradiction. Hence $\mathfrak{V} = \mathfrak{S}_E$ is true and we have $I/\mathfrak{V} = I_E/\mathfrak{S}_E$.

VIII $\Rightarrow$ VI. Take into account theorem 3 of [9] and the remark after theorem 3.

REFERENCES

[1] N. Bourbaki, *Intégration*, Chap. I-IV, Paris (1952).
[2] — *Intégration*, Chap. V, Paris (1956).
[3] — *Intégration*, Chap. VI, Paris (1959).
[4] S. Saks, *Theory of the integral*, Warsaw (1937).
[5] F. Terpe, *Zur deskriptiven Theorie der Integration*, Dissertation, Greifswald 1961.
[6] — *Maximale eingelagerte absolutkonvergente Integrale*, Mathematische Nachrichten 28 (1965), p. 257-274.
[7] — *Eine Charakterisierung des Lebesgue-Integrals als maximales eingelagertes absolutkonvergentes Integral*, Mathematische Nachrichten 32 (1966), p. 247-256.
[8] — *Theorie des maximalen Integrals*, Habilitationsschrift, Greifswald 1966.
[9] — *Das zu einem Maße gehörige Integral als maximales Integral*, Mathematische Nachrichten 41 (1969), p. 139-171.
[10] — *Das Stone-Integral als maximales eingelagertes absolutkonvergentes Integral*, Mathematische Zeitschrift 107 (1968), p. 59-66.
[11] — *Kompakte Mengen und Nichtfortsetzbarkeit von Integralen*, Proceedings of the I. International Symposium on Extension Theory of Topological Structures and its Applications, Berlin 1969.
[12] A. C. Zaanen, *A note on measure theory*, Nieuw Archiv voor Wiskunde (3) 6 (1958), p. 58-65.

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