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THE JOIN OF EQUATIONAL THEORIES

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It is known that the first-order theory of a single equivalence relation is decidable while the theory of two equivalence relations is not (1). This is an example of an interesting phenomenon in first-order logic: assuming we know all statements expressing properties of the individual primitive relations of a theory, there may still be no effective way of determining even the purely logical consequences of these properties. We shall show that this phenomenon does not occur in equational logic even though this logic is known to reflect much of the complexity of first-order logic.

More precisely, we shall prove that, for any two consistent equational theories Θ and Φ without common operation symbols or constants, the Turing degree of unsolvability of their join, that is, their smallest common extension, coincides with the join of the degrees of Θ and Φ in the semi-lattice of all degrees. In particular, the join is decidable just in case both Θ and Φ are decidable. We also show that an improvement of this result which naturally comes to mind actually fails to hold: theories which induce the same theory in the common part of their languages are constructed which are decidable but have an undecidable join.

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1. Preliminaries. We shall assume that the reader is familiar with the basic elements of equational logic as they are presented, for example, in the survey paper of Tarski [10]; furthermore, we assume familiarity with the notation and terminology used in [10].

In one minor departure from the terminology of [10] we shall assume that the rank of each operation symbol is somehow completely determined by the shape of the symbol so that any given system of equational logic

(1) The result for one equivalence relation was obtained, independently, by Janiczak [4] and Thompson [11]; for two relations, the result was independently obtained by Janiczak [4] and Rogers [8].

is completely determined by its (unordered) set Γ of operation symbols. It is this set Γ that we shall call the *type* of the system and it is then clear what we mean when we speak of terms, equations, and theories of type Γ. The sets of all terms, equations, and theories of type Γ are denoted by Te_Γ, Eq_Γ, and Th_Γ. The variables are common to all systems and are arranged in an infinite sequence $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_\varkappa, \ldots$ for $\varkappa < \omega$, where ω denotes the first infinite ordinal.

Given any theory Θ and terms τ and σ (all of the same type), we say that τ and σ are *equivalent in* Θ, or Θ-*equivalent*, or, simply, write $\tau \equiv_\Theta \sigma$, if the equation $\tau = \sigma$ is contained in Θ. If Θ and Φ are theories of type Γ and Δ, respectively, then by the *equational join* of Θ and Φ, in symbols $\Theta \vee \Phi$, we mean the theory of type $\Gamma \cup \Delta$ generated by $\Theta \cup \Phi$; formally, in the notation of [10],

$$\Theta \vee \Phi = \boldsymbol{\Theta}_{\Gamma \cup \Delta}[\Theta \cup \Phi].$$

Let τ be an arbitrary term and $\pi_0, \pi_1, \ldots, \pi_{\varkappa-1}$ specific occurrences of subterms in τ. It will be convenient to have a canonical procedure for replacing each π_μ in τ by a variable in order to obtain a new term σ with the property that τ can be reconstructed from σ by an ordinary substitution of terms for variables. The simplest way to do this is to introduce a new variable into our language for each π_μ; we now describe this process in more detail.

We assume first of all that there is a denumerable set Λ of operation symbols which is universal in the sense that it includes as subsets all of the types we shall consider; thus, every term we deal with is a member of Te_Λ. We assume further that there is a one-one function $\boldsymbol{v} = \langle \boldsymbol{v}\tau : \tau \in \mathrm{Te}_\Lambda \rangle$ which maps the universal set Te_Λ of terms onto an enlarged set Va^{e} of variable symbols. We assume that $\boldsymbol{v}\boldsymbol{x}_\varkappa = \boldsymbol{x}_\varkappa$ for each of the old variables $\boldsymbol{x}_\varkappa$ and that $\mathrm{Va}^{\mathrm{e}} \cap \mathrm{Te}_\Lambda = \mathrm{Va}$. It is now obvious how we can enlarge or extend each of our systems of equational logic to a new system with Va^{e} as its set of variables. With each type Γ we now associate two kinds of terms, *standard terms* and *extended terms*; we denote the set of extended terms by $\mathrm{Te}^{\mathrm{e}}_\Gamma$. Similar remarks apply in the case of equations and theories.

Let σ be any extended term and $\pi_0, \ldots, \pi_{\varkappa-1}$ a set of standard terms. Then, when we write

$$\tau = \sigma[\pi_0, \ldots, \pi_{\varkappa-1}], \tag{1}$$

we mean that τ is the result of simultaneous substituting π_λ for $\boldsymbol{v}\pi_\lambda$ in σ for each $\lambda < \varkappa$. Now, let τ be an arbitrary standard term, and let $\pi_0, \ldots, \pi_{\varkappa-1}$ be specific occurrences of subterms in τ such that, for distinct $\mu, \nu < \varkappa$, the occurrence of π_μ in τ does not lie within the occurrence of π_ν. Take σ to be the result of replacing each π_λ by $\boldsymbol{v}\pi_\lambda$. Then, σ is an extended term such that (1) holds. For example, if Q and P are binary operation symbols and τ is the term

$$QPx_0x_1QPx_0x_1Qx_0x_0, \tag{2}$$

π_0 is the first occurrence of Px_0x_1, and π_1 the only occurrence of Qx_0x_0, then σ becomes $QvPx_0x_1QPx_0x_1vQx_0x_1$.

By an *algebra of type* Γ we mean any algebra of the form

$$\mathfrak{A} = \langle A, Q^{(\mathfrak{A})}\rangle_{Q\in\Gamma}, \tag{3}$$

where $Q^{(\mathfrak{A})}$ is an operation on the universe A that has the same rank as Q. A capital German letter always denotes an algebra and the corresponding capital Latin letter the universe of the algebra. We express the fact that $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ (and hence of the same type as $\mathfrak{A}$) by writing $\mathfrak{B} \subseteq \mathfrak{A}$.

If $\Gamma' \subseteq \Gamma$ and $\mathfrak{A}$ is given by (3), then by the *Γ'-reduct* of $\mathfrak{A}$, in symbols $\mathfrak{Rd}_{\Gamma'}\mathfrak{A}$, we mean the algebra $\langle A, Q^{(\mathfrak{A})}\rangle_{Q\in\Gamma'}$ obtained by discarding operations corresponding to symbols in $\Gamma \sim \Gamma'$. If K and L are classes of algebras of type Γ and Δ, respectively, then the *equational meet* of K and L, in symbols $K \wedge L$, is defined to be the class of all algebras of type $\Gamma\cup\Delta$ such that $\mathfrak{Rd}_\Gamma\mathfrak{A} \in K$ and $\mathfrak{Rd}_\Delta\mathfrak{A} \in L$. Thus, if Θ and Φ are theories of type Γ and Δ, respectively, and K and L are the respective varieties of models of Θ and Φ, then $K\wedge L$ is the variety of $\Theta \vee \Phi$.

A theory is *consistent* if it does not contain every equation. Clearly, a theory is consistent iff it has a non-trivial model, i.e., a model with more than one element. Therefore, the consistency of $\Theta\vee\Phi$ implies the consistency of both Θ and Φ. The converse is not true in general but is true if $\Delta\cap\Gamma = 0$. It is easy to see that this latter condition, together with the consistency of $\Theta\vee\Phi$, also implies that $\Theta\vee\Phi$ is a *conservative extension* of both Θ and Φ in the sense that

$$(\Theta\vee\Phi)\cap \mathrm{Eq}_\Gamma = \Theta \quad \text{and} \quad (\Theta\vee\Phi)\cap\mathrm{Eq}_\Delta = \Phi;$$

again, this is not true in general.

If $\langle \mathfrak{A}_\varkappa : \varkappa < \omega\rangle$ is a sequence of algebras of type Γ such that $\mathfrak{A}_\varkappa \subseteq \mathfrak{A}_\lambda$ whenever $\varkappa \leqslant \lambda$, then the structure

$$\langle \bigcup_{\varkappa<\omega} A_\varkappa, \bigcup_{\varkappa<\omega} Q^{(\mathfrak{A}_\varkappa)}\rangle_{Q\in\Gamma}$$

turns out to be an algebra. It is called the *union* of the sequence and is written symbolically as $\bigcup_{\varkappa<\omega}\mathfrak{A}_\varkappa$. The union is thus a special case of direct limit.

Finally, we want to make a few remarks about concepts from the theory of recursive functions occurring in this paper. These notions prove to be quite useful in formulating our results concisely but no real use is made of deeper parts of the theory. Essentially, the only notion involved is that of relative recursiveness and this has a rather clear intuitive content:

a set N of natural numbers is *recursive* in another set M, or is *Turing reducible* to M, or the *Turing degree* of N is less than or equal to that of M, if there is an algorithm for computing the characteristic function of N that makes use of an "oracle" for the characteristic function of M. We shall rely heavily on Church's thesis to demonstrate relative recursiveness. The reader is referred to Rogers [8] for details.

We never deal directly with sets of natural numbers, of course, but rather with sets of terms, equations, ordered pairs of terms etc. When concerned with questions of effectivity involving such sets we shall automatically identify them with the corresponding sets of Gödel numbers.

We shall deal only with recursive sets of operation symbols. Thus the Te_Γ and Te^e_Γ considered will always be recursive. Moreover, there is assumed to exist a one-one recursive correspondence between Va and Va^e. This induces a recursive isomorphism between Te_Γ and Te^e_Γ which, in turn, induces a one-one correspondence between standard and extended theories. Corresponding theories are recursively isomorphic and for this reason we shall not bother to distinguish between them.

Given any pair of theories Θ and Φ, by the *recursive join* of Θ and Φ, in symbols Θ *join* Φ, we mean the set N *join* M defined on p. 81 of Rogers [8], where N and M are the respective sets of Gödel numbers corresponding to Θ and Φ. The important thing for our purposes is that the Turing degree of Θ *join* Φ is the join of the degrees of Θ and Φ.

2. The Turing degree of $\Theta \vee \Phi$. This section is devoted entirely to proving the following result:

THEOREM 1. *Let Γ and Δ be sets of operation symbols such that $\Gamma \cap \Delta = 0$. Assume $\Theta \in \mathrm{Th}_\Gamma$ and $\Phi \in \mathrm{Th}_\Delta$; in addition, assume that Θ and Φ are both consistent. Then $\Theta \vee \Phi$ is Turing equivalent to Θ join Φ. In particular, $\Theta \vee \Phi$ is decidable just in case both Θ and Φ are decidable.*

Since $\Gamma \cap \Delta = 0$ and $\Theta \vee \Phi$ is consistent, the latter is a conservative extension of both Θ and Φ. Hence, each of these theories is Turing reducible to their equational join, and it remains only to prove that $\Theta \vee \Phi$ is Turing reducible to Θ *join* Φ.

Throughout this section we assume that Γ, Δ, Θ and Φ are as described in the statement of the theorem. Furthermore, we take K and L to be the varieties of Θ and Φ, respectively. Notice that $K \wedge L$ is the variety of $\Theta \vee \Phi$.

We begin the proof by characterizing the structure of free algebras of $K \wedge L$ in terms of the free algebras of K and L; this characterization is due to Baranovič [1].

Let V be an arbitrary variety, $\mathfrak{A} \in V$, and X a set disjoint from A. An algebra $\mathfrak{B} \in V$ is called a *free extension in K of $\mathfrak{A}$ by X* whenever

(I) $\mathfrak{A} \subseteq \mathfrak{B}$ and $X \subseteq B$;

(II) $\mathfrak{B}$ is generated by $A \cup X$;

(III) given any $\mathfrak{C} \in V$, any homomorphism f from $\mathfrak{A}$ to $\mathfrak{C}$, and any function g from X into C, there exists a homomorphism h from $\mathfrak{B}$ into $\mathfrak{C}$ that extends both f and g.

A free extension is thus just a free product in K of $\mathfrak{A}$, and a free algebra generated by X. However, as opposed to the case for free products in general, free èxtensions always exist. For a discussion of free algebras and free products in the general theory of algebras see [2]; the terminology we use here, however, comes for the most part from [3].

We define simultaneously by recursion two sequences

$$\mathfrak{M} = \langle \mathfrak{M}_\varkappa : \varkappa < \omega \rangle \quad \text{and} \quad \mathfrak{N} = \langle \mathfrak{N}_\varkappa : \varkappa < \omega \rangle$$

of algebras of K and L, respectively. Choose algebras $\mathfrak{M}_0 \in K$ and $\mathfrak{N}_0 \in L$ such that both are freely generated by the set Va^{e}, and $M_0 \cap N_0 = \mathrm{Va}^{\mathrm{e}}$. For each $\varkappa < \omega$, choose $\mathfrak{M}_{\varkappa+1}$ to be a free extension in K of $\mathfrak{M}_\varkappa$ by $N_\varkappa \sim M_\varkappa$, and $\mathfrak{N}_{\varkappa+1}$ to be a free extension in L of $\mathfrak{N}_\varkappa$ by $M_\varkappa \sim N_\varkappa$; in addition, choose $\mathfrak{M}_{\varkappa+1}$ and $\mathfrak{N}_{\varkappa+1}$ such that

$$M_{\varkappa+1} \cap N_{\varkappa+1} = M_\varkappa \cup N_\varkappa. \tag{4}$$

Clearly, $\mathfrak{M}_\varkappa \subseteq \mathfrak{M}_\lambda$ and $\mathfrak{N}_\varkappa \subseteq \mathfrak{N}_\lambda$ whenever $\varkappa \leqslant \lambda < \omega$, so that the respective unions $\bigcup\limits_{\varkappa<\omega} \mathfrak{M}_\varkappa$ and $\bigcup\limits_{\varkappa<\omega} \mathfrak{N}_\varkappa$ exist. Moreover, in view of (4), these two algebras have the same universe. Let $\mathfrak{F}$ be the unique member of $K \wedge L$ such that

$$\mathfrak{Rd}_\Gamma \mathfrak{F} = \bigcup_{\varkappa<\omega} \mathfrak{M}_\varkappa \quad \text{and} \quad \mathfrak{Rd}_\Delta \mathfrak{F} = \bigcup_{\varkappa<\omega} \mathfrak{N}_\varkappa .$$

Let $\mathfrak{A}$ be any algebra in $K \wedge L$ and f an arbitrary function from Va^{e} into A. Because $\mathfrak{M}_\varkappa$ and $\mathfrak{N}_\varkappa$ are free extensions, it is obvious how to define, simultaneously, by recursion a pair of sequences of functions $\langle g_\varkappa : \varkappa < \omega \rangle$ and $\langle h_\varkappa : \varkappa < \omega \rangle$ with the following properties: for every $\varkappa < \omega$, $g_\varkappa$ is a homomorphism from $\mathfrak{M}_\varkappa$ into $\mathfrak{Rd}_\Gamma \mathfrak{A}$ and $h_\varkappa$ is a homomorphism from $\mathfrak{N}_\varkappa$ into $\mathfrak{Rd}_\Delta \mathfrak{A}$, and $g_{\varkappa+1} \cap h_{\varkappa+1} = g_\varkappa \cup h_\varkappa$. Furthermore, g_0 and h_0 are chosen so that $g_0 \cap h_0 = f$. Let

$$\bar{f} = \bigcup_{\varkappa<\omega} g_\varkappa = \bigcup_{\varkappa<\omega} h_\varkappa .$$

Then it is an easy matter to check that $\bar{f}$ is a homomorphism from $\mathfrak{F}$ into $\mathfrak{A}$ extending f. Thus we have

LEMMA 2. *$\mathfrak{F}$ is $(K \wedge L)$-freely generated by* Va^{e} (2).

Set

$$X = \mathrm{Va}^{\mathrm{e}} \cup \bigcup_{\varkappa<\omega} (N_\varkappa \sim M_\varkappa) \quad \text{and} \quad Y = \mathrm{Va}^{\mathrm{e}} \cup \bigcup_{\varkappa<\omega} (M_\varkappa \sim N_\varkappa).$$

(2) This lemma is a particular case of the lemma in Baranovič [1].

Then the following lemma is an immediate consequence of the construction of $\mathfrak{F}$:

LEMMA 3. (i) *$X \cup Y = F$ and $X \cap Y = \mathrm{Va}^{\mathrm{e}}$.*

(ii) *$\mathfrak{Rd}_{\Gamma}\mathfrak{F}$ is K-freely generated by X.*

(iii) *$\mathfrak{Rd}_{\Delta}\mathfrak{F}$ is L-freely generated by Y.*

Speaking very loosely, this lemma allows us to solve the word problem for $\mathfrak{F}$ under the assumption that we know how to solve the word problems for the free algebras over K and L; in view of Lemma 2, this is essentially the content of Theorem 1. In order to make this precise, however, we must relate the structure of compound terms, i.e., members of $\mathrm{Te}_{\Gamma\cup\Delta}$, to the structure of the algebra $\mathfrak{F}$ as described in Lemma 3. For this purpose it proves convenient to introduce some special terminology.

By the *principal operation symbol* of a non-variable term we mean the left-most symbol occurring in it; for instance, the principal operation symbol of (2) is Q. We let Te^*_{Γ} denote the set of all compound terms which are either variables or have their principal operation symbol in Γ; Te^*_{Δ} is defined analogously. Note that

$$\mathrm{Te}^*_{\Gamma} \cup \mathrm{Te}^*_{\Delta} = \mathrm{Te}_{\Gamma\cup\Delta} \quad \text{and} \quad \mathrm{Te}^*_{\Gamma} \cap \mathrm{Te}^*_{\Delta} = \mathrm{Va}.$$

Given any $\tau \in \mathrm{Te}^*_{\Gamma} \sim \mathrm{Va}$, τ can be expressed in the form

$$\tau = \sigma[\pi_0, \pi_1, \ldots, \pi_{\varkappa-1}], \tag{5}$$

where $\sigma \in \mathrm{Te}^{\mathrm{e}}_{\Gamma} \sim \mathrm{Va}^{\mathrm{e}}$ and $\pi_{\lambda} \in \mathrm{Te}^*_{\Delta}$ for each $\lambda < \varkappa$. The term σ is, moreover, uniquely determined; it is called the *principal part of* τ. The principal part is defined analogously when $\tau \in \mathrm{Te}^*_{\Delta} \sim \mathrm{Va}$.

Definition 4. A compound term σ is called *normal* if it contains no non-variable subterm whose principal part is equivalent in either Θ and Φ to a variable.

Suppose that ϱ contains a subterm $\tau \in \mathrm{Te}^*_{\Gamma}$ of form (5) and that the principal part σ of τ is Θ-equivalent to the variable $v\pi_{\lambda}$ for some $\lambda < \varkappa$. By replacing the occurrence of τ in ϱ by π_{λ}, we obtain a new term ϱ' equivalent in $\Theta \vee \Phi$ to ϱ but shorter in length. By making such replacements in a fixed uniform way, we arrive after a finite number of steps at a normal term ϱ'' that is $(\Theta \vee \Phi)$-equivalent to ϱ. This proves

LEMMA 5. *There exists a function φ from $\mathrm{Te}_{\Gamma\cup\Delta}$ into the set of normal terms such that φ is Turing reducible to Θ join Φ and $\varphi\tau \equiv_{\Theta\vee\Phi} \tau$ for every $\tau \in \mathrm{Te}_{\Gamma\cup\Delta}$.*

Thus, in our search for an algorithm for reducing the decision problem for $\Theta \vee \Phi$ to that of Θ *join* Φ, we can restrict our attention to normal terms.

Let $\boldsymbol{h}$ be the unique function from the set $\mathrm{Te}^{\mathrm{e}}_{\Gamma\cup\Delta}$ of all extended compound terms into the universe F of $\mathfrak{F}$ that is defined by recursion on the length of terms in the following way: $\boldsymbol{h}v = v$ for every $v \in \mathrm{Va}^{\mathrm{e}}$ and, if $\tau = Q\pi_0 \ldots \pi_{\varkappa-1} \in \mathrm{Te}^{\mathrm{e}}_{\Gamma \cup \Delta} \sim \mathrm{Va}^{\mathrm{e}}$,

$$\boldsymbol{h}\tau = Q^{(\mathfrak{F})}(\boldsymbol{h}\pi_0, \ldots, \boldsymbol{h}\pi_{\varkappa-1}).$$

Alternatively, $\boldsymbol{h}$ can be defined as the unique homomorphism from the algebra of extended terms (3) of type $\Gamma \cup \Delta$ into $\mathfrak{F}$ which reduces to the identity function on Va^{e}. Hence, since $\mathfrak{F}$ is $(K \wedge L)$-freely generated by Va^{e}, for any $\sigma, \sigma' \in \mathrm{Te}^{\mathrm{e}}_{\Delta \cup \Gamma}$, we have

(6) $$\sigma \equiv_{\Theta \vee \Phi} \sigma' \quad \text{iff} \quad \boldsymbol{h}\sigma = \boldsymbol{h}\sigma'.$$

LEMMA 6. *Let τ be a normal term in* $\mathrm{Te}_{\Gamma \cup \Delta}$.

(i) $\boldsymbol{h}\tau \in Y$ *for every* $\tau \in \mathrm{Te}^*_{\Gamma}$.

(ii) $\boldsymbol{h}\tau \in X$ *for every* $\tau \in \mathrm{Te}^*_{\Delta}$.

(iii) $\boldsymbol{h}\tau \in \mathrm{Va}^{\mathrm{e}}$ *iff* $\tau \in \mathrm{Va}$.

Proof. (i) and (ii) are proved simultaneously by induction on the length of τ. If τ is a variable, then $\boldsymbol{h}\tau = \tau \in \mathrm{Va} \subseteq X \cap Y$. If τ is a constant in Γ, then $\boldsymbol{h}\tau \in M_0 \sim N_0 \subseteq Y$, and, if τ is a constant in Δ, then $\boldsymbol{h}\tau \in N_0 \sim M_0 \subseteq X$. Assume, therefore, that $|\tau| > 1$. We can also assume that $\tau \in \mathrm{Te}^*_{\Gamma}$ since the case $\tau \in \mathrm{Te}^*_{\Delta}$ will follow by symmetry. Write τ in form (5), where σ is the principal part of τ and $\pi_\mu \in \mathrm{Te}^*_{\Delta}$ for each $\mu < \varkappa$. We then have

(7) $$\boldsymbol{h}\tau = \tilde{\sigma}(\boldsymbol{h}\pi_0, \ldots, \boldsymbol{h}\pi_{\varkappa-1}),$$

where $\tilde{\sigma}$ is the polynomial in $\varkappa$ variables over $\mathfrak{Rd}_{\Gamma}\mathfrak{F}$ that is determined by σ; cf. [3], p. 43 and Theorem 0.4.60. Observe that the terms π_μ are all normal and of length less than $|\tau|$. Thus, by the induction hypothesis,

(8) $$\boldsymbol{h}\pi_\mu \in X \quad \text{for each } \mu < \varkappa.$$

If $\boldsymbol{h}\tau \notin Y$, then by Lemma 3, (i), we must have

(9) $$\boldsymbol{h}\tau \in X.$$

But $\mathfrak{Rd}_{\Gamma}\mathfrak{F}$ is K-freely generated by X (cf. Lemma 3, (ii)); hence, in view of (7)-(9), σ must be Θ-equivalent to a variable and this contradicts the normality of τ. Thus $\boldsymbol{h}\tau \in Y$, and parts (i) and (ii) are proved. Part (iii) is proved similarly.

We are now in a position to complete the proof of Theorem 1.

Consider an arbitrary pair τ, τ' of normal compound terms. Suppose one of the terms, say τ, is a variable. If τ' is not a variable, then, by Lemma 6, (ii), $\boldsymbol{h}\tau \neq \boldsymbol{h}\tau'$ so that τ is not $(\Theta \vee \Phi)$-equivalent to τ'. If τ' is a variable, then, since $\Theta \vee \Phi$ is consistent, τ, τ' are $(\Theta \vee \Phi)$-equivalent just in case

(3) For the definition of this algebra and a discussion of its relation to free algebras see [3], p. 143 ff.

they are identical. Therefore, we may assume that neither τ nor τ' is a variable. Express the terms in the forms

$$\tau = \sigma[\pi_0, \ldots, \pi_{\varkappa-1}] \quad \text{and} \quad \tau' = \sigma'[\pi_\varkappa, \ldots, \pi_{\varkappa+\lambda-1}],$$

where σ, σ' are the principal parts of τ, τ', respectively. We have τ and τ' $(\Theta \vee \Phi)$-equivalent iff

$$h\tau = h\tau'. \tag{10}$$

By Lemma 3, (i), and Lemma 6, equality (10) holds only in case both $\tau, \tau' \in \mathrm{Te}_\Gamma^*$ or both $\tau, \tau' \in \mathrm{Te}_\Delta^*$. We assume without loss of generality that $\tau, \tau' \in \mathrm{Te}_\Gamma^*$. Under this assumption equality (10) becomes equivalent to

$$\tilde{\sigma}(h\pi_0, \ldots, h\pi_{\varkappa-1}) = \tilde{\sigma}'(h\pi_\varkappa, \ldots, h\pi_{\varkappa+\lambda-1}), \tag{11}$$

where $\tilde{\sigma}, \tilde{\sigma}'$ are the polynomials over $\mathfrak{Rd}_\Gamma\mathfrak{F}$ determined by σ, σ'. Since $\tau, \tau' \in \mathrm{Te}_\Gamma^*$, we have $\pi_\mu \in \mathrm{Te}_\Delta^*$ and thus, by Lemma 6, (i), $h\pi_\mu \in X$ for each $\mu < \varkappa + \lambda$. Hence, applying Lemma 3, (ii), we conclude that (11) holds just in case there exists a $\xi < \omega$ and a function u from the $\varkappa + \lambda$ terms $\pi_0, \ldots, \pi_{\varkappa+\lambda-1}$ onto the first ξ variables $x_0, \ldots, x_{\xi-1}$ such that, for all μ, $\nu < \varkappa + \lambda$,

$$\pi_\mu \equiv_{\Theta\vee\Phi} \pi_\nu \quad \text{when } u\pi_\mu = u\pi_\nu,$$
$$\pi_\mu \not\equiv_{\Theta\vee\Phi} \pi_\nu \quad \text{when } u\pi_\mu \neq u\pi_\nu,$$

and

$$\bar{\sigma} \equiv_\Theta \bar{\sigma}',$$

where $\bar{\sigma}, \bar{\sigma}'$ are obtained from σ, σ' by substituting $u\pi_\eta$ for $v\pi_\eta$ for each $\eta < \varkappa + \lambda$. Hence, the question whether τ and τ' are $(\Theta \vee \Phi)$-equivalent reduces in a uniform way to a finite Boolean combination of questions about the equivalence of terms in Θ (or Φ) and questions about the equivalence of terms in $\Theta \vee \Phi$ which are of length less than the maximum of $|\tau|$ and $|\tau'|$. Therefore, $\Theta \vee \Phi$ is Turing reducible to Θ *join* Φ and the proof of Theorem 1 is complete.

Something stronger has been proved than what was stated in Theorem 1. An analysis of the proof just given shows that $\Theta \vee \Phi$ is actually (unbounded) truth-table equivalent to Θ *join* Φ.

3. An example. The question naturally arises as to what extent the premise $\Gamma \cap \Delta = 0$ in the statement of Theorem 1 can be weakened without destroying the validity of the theorem. It is not difficult to see that it cannot be completely eliminated, but Alfred Tarski suggested that it might be possible to replace it by the equality $\Theta \cap \mathrm{Eq}_{\Gamma\cap\Delta} = \Phi \cap \mathrm{Eq}_{\Gamma\cap\Delta}$. Actually, we shall show in Theorem 8 below that this condition does not even guarantee that the join of decidable theories is decidable. In order to establish this result we require the following

LEMMA 7. *Assume Γ is a set of operation symbols. Let $\sigma_0, \sigma_1, \ldots, \sigma_{\varkappa-1}$ and $\tau_0, \tau_1, \ldots, \tau_{\varkappa-1}$ be any pair of sequences of terms in* $\mathrm{Te}_\Gamma \sim \mathrm{Va}$ *and* Te_Γ, *respectively, such that the following condition is satisfied*:

for each $\lambda < \varkappa$, the principal operation symbol of σ_λ does not occur in any proper subterm of σ_λ nor in any of the terms $\sigma_0, \ldots, \sigma_{\lambda-1}, \sigma_{\lambda+1}, \ldots, \sigma_{\varkappa-1}$ and $\tau_0, \ldots, \tau_\lambda$.

Then the theory

$$\boldsymbol{\Theta}[\{\sigma_0 = \tau_0, \ldots, \sigma_{\varkappa-1} = \tau_{\varkappa-1}\}]$$

is decidable.

This lemma is an immediate consequence of the corollary of Theorem 5, p. 274, of [5]. In the terminology of [5], for all σ_μ and σ_ν, the superposition of σ_ν on a non-variable subterm π of σ_μ exists only in case $\mu = \nu$ and $\pi = \sigma_\nu$. Thus the set of equations $\sigma_0 = \tau_0, \ldots, \sigma_{\varkappa-1} = \tau_{\varkappa-1}$ considered as reductions form a complete set. From this fact it follows that they must generate a decidable theory.

THEOREM 8. *There exist sets Γ and Δ of operation symbols and finitely based theories $\Theta \in \mathrm{Th}_\Gamma$ and $\Phi \in \mathrm{Th}_\Delta$ such that Θ and Φ are decidable and $\Theta \cap \mathrm{Eq}_{\Gamma\cap\Delta} = \Phi \cap \mathrm{Eq}_{\Gamma\cap\Delta}$, but $\Theta \vee \Phi$ fails to be decidable.*

Proof. It has been shown by both Mal'cev [6] and Perkins [7] that there exists a finitely based undecidable theory with a single binary operation symbol. Let M be such a theory. Let N be the theory of rings with unit where addition, multiplication, additive inverse, and the multiplicative unit are taken to be the fundamental operations. Finally, let Ψ be the equational join of M and N. By Theorem 1, Ψ is undecidable and, by Theorem 4 of Tarski [10], Ψ has a base consisting of a single equation $\tau = \sigma$. It is easy to see that either τ or σ must be a variable, so assume that σ is the variable $\boldsymbol{x}_0$. Since Ψ is consistent, $\boldsymbol{x}_0$ occurs in τ. The result of substituting the term $\varphi \boldsymbol{x}_\varkappa$ for each $\boldsymbol{x}_\varkappa$ in τ is denoted by $\mathrm{Su}_\varphi \tau$.

Let K be the set of operation symbols of Ψ and keep in mind that $\Psi = \boldsymbol{\Theta}_K[\{\tau = \boldsymbol{x}_0\}]$, and Ψ is undecidable. Let H, Q_0, Q_1 be distinct unary operation symbols not contained in K, and let

$$\Theta = \boldsymbol{\Theta}_\Gamma[\{H\tau = \boldsymbol{x}_0, Q_0\boldsymbol{x}_0 = HQ_1\boldsymbol{x}_0\}], \quad \text{where } \Gamma = K \cup \{H, Q_0, Q_1\}.$$

Finally, let L be a unary operation symbol not contained in Γ, and let

$$\Phi = \boldsymbol{\Theta}_\Delta[\{Q_0L\boldsymbol{x}_0 = \boldsymbol{x}_0, Q_1L\boldsymbol{x}_0 = \boldsymbol{x}_0\}], \quad \text{where } \Delta = \{Q_0, Q_1, L\}.$$

It follows immediately from Lemma 7 that the theories Θ and Φ are decidable. On the other hand, we have

$$H\boldsymbol{x}_0 \equiv_{\Theta\vee\Phi} HQ_1L\boldsymbol{x}_0 \equiv_{\Theta\vee\Phi} Q_0L\boldsymbol{x}_0 \equiv_{\Theta\vee\Phi} \boldsymbol{x}_0,$$

$$\tau \equiv_{\Theta\vee\Phi} H\tau \equiv_{\Theta\vee\Phi} \boldsymbol{x}_0, \qquad Q_0\boldsymbol{x}_0 \equiv_{\Theta\vee\Phi} HQ_1\boldsymbol{x}_0 \equiv_{\Theta\vee\Phi} Q_1\boldsymbol{x}_0.$$

Thus $(Hx_0 = x_0)$, $(\tau = x_0)$, $(Q_0x_0 = Q_1x_0) \in \Theta \vee \Phi$ and it is easy to see that these three equations together with $Q_0Lx_0 = x_0$ form a base for $\Theta \vee \Phi$. Therefore, $(\Theta \vee \Phi) \cap \mathrm{Eq}_K = \Psi$, and hence $\Theta \vee \Phi$ is undecidable since Ψ is.

To complete the proof it only remains to show that

(12) $$\Theta \cap \mathrm{Eq}_{\Gamma \cap \Delta} = \{\tau \rightleftharpoons \tau : \tau \in \mathrm{Te}_{\Gamma \cap \Delta}\} = \Phi \cap \mathrm{Eq}_{\Gamma \cap \Delta}.$$

To prove these equalities we employ two simple model-theoretical arguments. Let $\mathfrak{A} = \langle A, q_0, q_1 \rangle$ be any algebra with two unary operations q_0 and q_1 interpreting the operation symbols Q_0 and Q_1, respectively. Let $A' = \omega \times A$ and identify A with $\{0\} \times A$ in the obvious way. We extend q_0 and q_1 to functions q_0' and q_1' on A' by taking

$$q_0'(\langle \varkappa + 1, a \rangle) = q_1'(\langle \varkappa + 1, a \rangle) = \langle \varkappa, a \rangle \quad \text{for each } \varkappa < \omega \text{ and } a \in A.$$

Let l be the transformation of A' such that $l(\langle \varkappa, a \rangle) = \langle \varkappa + 1, a \rangle$ for each $\langle \varkappa, a \rangle \in A'$. Then it is clear that $\mathfrak{A}' = \langle A', q_0', q_1', l \rangle$ is a model of Φ, and, since $\mathfrak{A}$ is a subalgebra of the reduct $\langle A, q_0', q_1' \rangle$ of $\mathfrak{A}$, we see that every equation involving only Q_0 and Q_1 that fails to hold in $\mathfrak{A}$ also fails in $\mathfrak{A}'$. Since $\mathfrak{A}$ is an arbitrary algebra with two unary operations, it follows that the second equality of (12) holds.

To demonstrate the first equality let $A = \mathrm{Te}_{K \cup \{Q_0, Q_1\}}$. For each $P \in K$, let p be the operation on A of the same rank as P such that $p(\varrho_0, \ldots, \varrho_{\lambda-1}) = P\varrho_0 \ldots \varrho_{\lambda-1}$ for all $\varrho_0, \ldots, \varrho_{\lambda-1} \in A$; let q_0 and q_1 be the unary operations on A such that $q_0(\varrho) = Q_0\varrho$ and $q_1(\varrho) = Q_1\varrho$ for every $\varrho \in A$. Finally, let h be any unary operation on A such that $h(\mathrm{Su}_\varphi \tau) = \varphi x_0$ for every function φ from Va into A, and $h(Q_1\varrho) = Q_0\varrho$ for every $\varrho \in A$. Notice that, because x_0 occurs in τ and Q_1 is not the principal operation symbol of τ, these two conditions are consistent, so that an h satisfying them exists. Clearly, the algebra $\langle A, p, q_0, q_1, h \rangle_{P \in K}$ is a model of Θ in which an equation $\varepsilon \in \mathrm{Eq}_{\Gamma \cap \Delta}$ holds just in case ε is a tautology. This gives the first equality of (12) and completes the proof of the theorem.

The equational join of the theories Θ and Φ constructed in the proof of Theorem 8 is not a conservative extension of either Θ or Φ, and it would be interesting to find theories Θ and Φ satisfying all the conditions of Theorem 8 such that this is the case. (**P 890**) It seems to us that such theories must certainly exits but their construction appears to present some difficulty.

For example, in view of the first paragraph of the introductory remarks, a likely candidate for such a pair of theories might seem to be the following: take both Θ and Φ to be the equational theory of the cylindric algebra of formulas that is associated with the first-order theory of a single equivalence relation. (Cf. [3] for definitions of the various notions from the theory of cylindric algebras involved here.) The set Γ of operation symbols of Θ will then consist of the usual fundamental operation symbols of the theory of cylindric algebras together with a single constant symbol representing the primitive equivalence relation. The set Δ of operation symbols of Φ is the same except that the primitive equivalence relation is represented by a different constant symbol. It is obvious that the theories Θ and Φ induce the same theory in the common part of their language and that their equational join is a conservative extension of both of them. Furthermore, the join is undecidable since it is the theory of the cylindric algebra of formulas associated with the first-order theory of two equivalence relations. However, it remains an open problem whether or not the theories Θ and Φ are decidable. (**P 891**) From the undecidability of the first-order theory of a single equivalence relation we can only conclude that there is a decision procedure for the equations of Θ and Φ which contain no variables.

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