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A CHARACTERIZATION OF COMPLETE BI-BROUWERIAN LATTICES

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1. In the paper [1] Abian proved that a Boolean ring is complete if and only if any 2-solvable system of one-variable Boolean polynomial equations is solvable.

The purpose of this note is to characterize those lattices in which every 2-solvable system of one-variable lattice polynomial equations is solvable.

The main result is the following

THEOREM. *A lattice has the property that any 2-solvable system of one-variable lattice equations is solvable if and only if it is complete and bi-Brouwerian.*

2. *One-variable lattice polynomials* are defined inductively as follows:

(i) $p_0 = a$, $p_1 = x$ are lattice polynomials.

(ii) If p, q are lattice polynomials in one variable x, say, then so are $p \vee q$, $p \wedge q$.

A *polynomial equation in x over a lattice $\mathscr{L}$* is an expression of the form $p = q$, where p and q are one-variable lattice polynomials in x over $\mathscr{L}$.

A set Σ of polynomial equations in x is said to be 2-*solvable* if every subsystem of Σ consisting of two equations is solvable.

A *Brouwerian lattice* is a lattice $\mathscr{L}$ in which, for every $a, b \in \mathscr{L}$, there exists $a * b \in \mathscr{L}$ such that

$$a \wedge x \leqslant b \Leftrightarrow x \leqslant a * b.$$

A *bi-Brouwerian lattice* is a Brouwerian lattice whose dual is Brouwerian, i.e., for every $a, b \in \mathscr{L}$, there exists $a - b \in \mathscr{L}$ such that

$$a \vee x \geqslant b \Leftrightarrow x \geqslant a - b.$$

Any Brouwerian lattice is distributive.

3. Let S be any non-empty subset of a lattice $\mathscr{L}$ having the property that any system of 2-solvable lattice polynomial equations in one variable is solvable.

Consider the system Σ_0 of equations $s \wedge x = s$ $(s \in S)$. Clearly, every 2-equation subsystem $s_1 \wedge x = s_1$ and $s_2 \wedge x = s_2$ has the solution $x = s_1 \vee s_2$ and so Σ_0 is solvable; any solution being an upper bound for S. Let U_0 be the set of all solutions of Σ_0 and consider the system Σ_1 of equations

$$s \wedge x = s \qquad (s \in S),$$

$$u \vee x = u \qquad (u \in U_0).$$

Clearly, a 2-equation subsystem of the form

$$s_1 \wedge x = s_1,$$

$$u_1 \vee x = u_1$$

has the solution $x = u_1$, while a 2-equation subsystem of the form

$$u_1 \vee x = u_1,$$

$$u_2 \vee x = u_2$$

has the solution $x = u_1 \wedge u_2$.

Consequently, Σ_1 is 2-solvable and, therefore, Σ_1 has a solution which is, obviously, the least upper bound of S. Hence, $\mathscr{L}$ is join-complete and, specializing to the case where $S = \mathscr{L}$, has a greatest element 1. Similarly, we can show that $\mathscr{L}$ is meet-complete and has a least element 0. Now, let $a, b \in \mathscr{L}$, $U_{ab} = \{x \in \mathscr{L};\, a \wedge x \leqslant b\}$ and consider the system Σ_2 of equations

$$b \vee (a \wedge x) = b,$$

$$u \wedge x = u \qquad (u \in U_{ab}).$$

Clearly, a 2-equation subsystem of the form $b \vee (a \wedge x) = b$ and $u \wedge x = u$ has the solution $x = u$, while a 2-equation subsystem of the form $u_1 \wedge x = u_1$ and $u_2 \wedge x = u_2$ has the solution $x = u_1 \vee u_2$.

Consequently, Σ_2 is 2-solvable and, therefore, Σ_2 has a solution which is the largest solution of the inequality $a \wedge x \leqslant b$. Thus, $\mathscr{L}$ is Brouwerian. Similarly, the dual of $\mathscr{L}$ is Brouwerian so that $\mathscr{L}$ is bi-Brouwerian.

4. In preparation for the proof of the converse we state

LEMMA 1. *In a bi-Brouwerian lattice, the following, together with their duals, hold:*

(i) $x \leqslant y \Leftrightarrow x * y = 1$,

(ii) $y \leqslant x * y$,

(iii) $x * (y * z) = (x \wedge y) * z$,

(iv) $x*(y\wedge z) = (x*y)\wedge(x*z)$,

(v) $(x\vee y)*z = (x*z)\wedge(y*z)$.

Proof. All of these results are proved in [3] except (iii) which is proved in [5].

LEMMA 2. *A complete lattice $\mathscr{L}$ is Brouwerian if and only if*

$$a\wedge\bigvee_{a} a_a = \bigvee_{a}(a\wedge a_a)$$

for any non-empty subset $\{a_a\}_{a\in I}$ of $\mathscr{L}$.

LEMMA 3. *Any one-variable lattice polynomial $p(x)$ in a distributive lattice with 0 and 1 can be uniquely expressed in the normal form*

$$p(x) = p_0\vee(p_1\wedge x) \quad \textit{with } p_0\leqslant p_1.$$

The proofs of Lemmas 2 and 3 are well known and may be found in [2] and [4], respectively.

If $a\leqslant b$ in $\mathscr{L}$, then the *interval* $[a, b]$ is the set $\{x\in\mathscr{L};\, a\leqslant x\leqslant b\}$.

In a Brouwerian lattice we write $a\times b$ for $(a*b)\wedge(b*a)$, and in a lattice whose dual is Brouwerian we write $a+b$ for $(a-b)\vee(b-a)$.

The following result is crucial:

LEMMA 4. *If $\mathscr{L} = \langle L; \vee, \wedge, *, -; 0, 1\rangle$ is a bi-Brouwerian lattice, then necessary and sufficient conditions for the existence of a solution of the pair Σ of lattice polynomial equations $p(x) = q(x)$ and $s(x) = t(x)$, where p, q, s, t are in the normal form, are the following:*

$$(*) \qquad q_0\leqslant p_1,\ p_0\leqslant q_1,\ t_0\leqslant s_1,\ s_0\leqslant t_1$$

and

$$p_0+q_0\leqslant s_1\times t_1,\ s_0+t_0\leqslant p_1\times q_1.$$

Moreover, if the pair Σ is solvable, then the solution set is the interval

$$[(p_0+q_0)\vee(s_0+t_0),\ (p_1\times q_1)\wedge(s_1\times t_1)].$$

Proof. For both equalities to hold it is necessary and sufficient that

$$\{p(x)+q(x)\}\vee\{s(x)+t(x)\} = 0.$$

Now, by the distributivity of $\mathscr{L}$,

$$\begin{aligned} p(x)-q(x) = 0 &\Leftrightarrow \{p_1\wedge(p_0\vee x)\}-q(x) = 0\\ &\Leftrightarrow \{p_1-q(x)\}\vee\{(p_0\vee x)-q(x)\} = 0\\ &\Leftrightarrow q(x)\leqslant p_1 \text{ and } q(x)\leqslant p_0\vee x.\end{aligned}$$

However,

$$\begin{aligned} q(x)\leqslant p_1 &\Leftrightarrow q_1\wedge(q_0\vee x)\leqslant p_1\\ &\Leftrightarrow q_0\vee x\leqslant q_1*p_1\\ &\Leftrightarrow q_0\leqslant p_1 \text{ and } x\leqslant q_1*p_1.\end{aligned}$$

Furthermore,

$$\begin{aligned} q(x) \leqslant p_0 \vee x &\Leftrightarrow p_0 - q(x) \leqslant x \\ &\Leftrightarrow p_0 - \{q_0 \vee (q_1 \wedge x)\} \leqslant x \\ &\Leftrightarrow (p_0 - q_0) \vee \{p_0 - (q_1 \wedge x)\} \leqslant x \\ &\Leftrightarrow p_0 - q_0 \leqslant x \text{ and } p_0 - (q_1 \wedge x) \leqslant x \end{aligned}$$

which, since $p_0 - (q_1 \wedge x) \leqslant q_1 \wedge x \leqslant x$, is equivalent to $p_0 - q_0 \leqslant x$. Consequently,

$$p(x) - q(x) = 0 \Leftrightarrow q_0 \leqslant p_1 \text{ and } p_0 - q_0 \leqslant x \leqslant q_1 * p_1,$$

and, therefore,

$$p(x) + q(x) = 0 \Leftrightarrow q_0 \leqslant p_1, p_0 \leqslant q_1 \text{ and } p_0 + q_0 \leqslant x \leqslant p_1 \times q_1.$$

Hence, $p(x) = q(x)$ and $s(x) = t(x)$ if and only if conditions $(*)$ hold and

$$(p_0 + q_0) \vee (s_0 + t_0) \leqslant x \leqslant (p_1 \times q_1) \wedge (s_1 \times t_1).$$

Finally, if conditions $(*)$ hold, then

$$p_0 - q_0 \leqslant q_0 \leqslant p_1 \leqslant q_1 * p_1 \quad \text{and} \quad p_0 - q_0 \leqslant q_0 \leqslant q_1 \leqslant p_1 * q_1$$

so that $p_0 - q_0 \leqslant p_1 \times q_1$.

Similarly, $q_0 - p_0 \leqslant p_1 \times q_1$ and, consequently, $p_0 + q_0 \leqslant p_1 \times q_1$. In a similar fashion we can show that $s_0 + t_0 \leqslant s_1 \times t_1$. Therefore,

$$(p_0 + q_0) \vee (s_0 + t_0) \leqslant (p_1 \times q_1) \wedge (s_1 \times t_1) \Leftrightarrow p_0 + q_0 \leqslant s_1 \times t_1$$

and

$$s_0 + t_0 \leqslant p_1 \times q_1,$$

completing the proof of Lemma 4.

Now we are in a position to prove the sufficiency.

Let $\mathscr{L}$ be a complete bi-Brouwerian lattice and let Σ: $p_i(x) = q_i(x)$, $i \in I$, be a system of polynomial equations, in one variable, which is 2-solvable. We may suppose, since $\mathscr{L}$ is distributive, that each p_i is in its normal form

$$p_i(x) = p_{i0} \vee (p_{i1} \wedge x) \qquad \text{with } p_{i0} \leqslant p_{i1}.$$

Now, if j is an arbitrary member of the index set I, then, since Σ is 2-solvable, it follows from Lemma 4 that $p_j(x_i) = q_j(x_i)$ for each $i \in I$, where

$$x_i = (p_{j0} + q_{j0}) \vee (p_{i0} + q_{i0}).$$

We deduce that

$$p_{j0} \vee \bigvee_{i \in I} (p_{j1} \wedge x_i) = q_{j0} \vee \bigvee_{i \in I} (q_{j1} \wedge x_i)$$

or, equivalently, from Lemma 2,

$$p_{j0} \vee (p_{j1} \wedge \bigvee_{i \in I} x_i) = q_{j0} \vee (q_{j1} \wedge \bigvee_{i \in I} x_i).$$

Consequently, the equation $p_j(x) = q_j(x)$ has the solution

$$x = \bigvee_{i \in I} x_i = \bigvee_{i \in I} (p_{i0} + q_{i0})$$

and, therefore, since j is arbitrary, it follows that Σ has a solution, namely

$$x = \bigvee_{i \in I} (p_{i0} + q_{i0}).$$

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