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SOME EXAMPLES OF IRREDUCIBLY CONFLUENT MAPPINGS

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We give four examples of confluent mappings two of which imply answers to questions asked in [6] (P 956 and P 957). The topological spaces under consideration are assumed to be metric, and the mappings — to be continuous and surjective. A mapping f of a topological space X onto a topological space Y is said to be

(i) *monotone* if, for any continuum Q in Y, the set $f^{-1}(Q)$ is connected (see [4], p. 131);

(ii) *open* if it transforms open sets into open sets;

(iii) *quasi-interior* if the conditions $y \in Y$, C is a component of $f^{-1}(y)$, and U is an open set containing C imply $y \in \operatorname{Int} f(U)$ (see [5]);

(iv) *confluent* if, for every subcontinuum Q of Y, each component of the inverse image $f^{-1}(Q)$ is mapped by f onto Q (see [2], p. 213);

(v) *light* if $\dim f^{-1}(y) = 0$ for each $y \in Y$ (see [7], p. 130).

It is proved (see [5], Corollary 3.1) that a mapping f is quasi-interior if and only if $f: X \to Y$ can be represented as a composition of mappings f_1 and f_2, $f = f_2 f_1$ (which means that $f(x) = f_2(f_1(x))$ for each $x \in X$), where f_1 is monotone and f_2 is open. Therefore, any monotone mapping is quasi-interior, and so is any open mapping. Moreover, every quasi-interior mapping is confluent (see [5], Corollary 2.7, p. 103).

A confluent (quasi-interior) mapping f from X onto Y is said to be *irreducibly confluent* (*irreducibly quasi-interior*) if there exists no proper subcontinuum H of X such that $f|H$ is a confluent (quasi-interior) mapping of H onto Y (see [6], p. 49). If f is a mapping of a continuum X onto a continuum Y such that f maps no proper subcontinuum of X onto Y, then f is called *irreducible*.

Recall that a continuum is said to be *unicoherent* if for any decomposition into two subcontinua the intersection of those subcontinua is connected. A continuum is called *hereditarily unicoherent* provided each of its subcontinua is unicoherent. A *dendrite* is a hereditarily unicoherent, locally connected continuum. A *dendroid* is a hereditarily unicoherent arcwise connected continuum.

Read asked the following question in [6], p. 51 (P 956):

Is it true that if f is a quasi-interior mapping of a hereditarily unicoherent continuum X onto Y, then there is a subcontinuum L of X such that $f|L$ is an irreducibly quasi-interior mapping of L onto Y?

The answer is negative and it follows from the following

Example 1. Let (r, φ, t) denote a point of the Euclidean 3-space having r, φ and t as its cylindrical coordinates. For each $n = 1, 2, \ldots$ put

$$p = (0, 0, 0), \quad q = (1, 0, 0), \quad q' = (1, \pi, 0),$$
$$q_n = (1, \pi/2n, 0), \quad q'_n = (1, \pi+\pi/2n, 0),$$
$$p_n = (1, 0, 1/n), \quad p'_n = (1, \pi, 1/n).$$

We denote the straight-line interval joining points a and b by $I(a, b)$. Using the above notation we write

$$X = I(q, q') \cup \bigcup_{n=1}^{\infty} \big(I(q_n, q') \cup I(q'_n, q) \cup I(p_n, p) \cup I(p'_n, p)\big).$$

Obviously, X is a dendroid. We define a mapping f of X onto $f(X)$ as follows:

$$f(r, \varphi, t) = \begin{cases} (r, 2\varphi, t) & \text{if } 0 \leqslant \varphi \leqslant \pi, \\ \big(r, 2(\varphi-\pi), t\big) & \text{if } \pi \leqslant \varphi \leqslant 2\pi. \end{cases}$$

It is easy to see that f is an open mapping. If L is a subcontinuum of X such that $f|L$ is a quasi-interior mapping of L onto $Y = f(X)$, then for each $n = 1, 2, \ldots$ either $q_n \in L$ or $q'_n \in L$, since $f^{-1}f(q_n) = \{q_n, q'_n\}$. Therefore, $\{q, q'\} \subset L$. Thus there must exist a positive integer m_0 such that if $m > m_0$, then $\{p_m, p'_m\} \subset L$. But

$$L_1 = \big(L \setminus I(p, p_{m_0+1})\big) \cup \{p\}$$

is a proper subcontinuum of L such that $f|L_1$ is a quasi-interior mapping of L_1 onto Y. Hence, there exists no subcontinuum L of X such that $f|L$ is an irreducibly quasi-interior mapping of L onto Y.

It is known (see [7], Theorem 2.4, p. 188) that if f is an open light mapping of a compact space X onto a dendrite Y, then there is a subcontinuum L of X such that f restricted to L is a homeomorphism of L onto Y. One can ask if such an implication is true for dendroids. The answer is negative, which can be seen from Example 1. Such a counterexample may be realized also in the plane. We have the following

Example 2. Let (r, φ) denote a point of the Euclidean plane having r and φ as its polar coordinates. Put

$$p = (0, 0), \quad q = (1, 0), \quad q' = (1, \pi),$$
$$q_n = (1, \pi/2n), \quad q'_n = (1, \pi+\pi/2n), \quad p_n = (1/2n, 0), \quad p'_n = (1/2n, \pi).$$

We denote the straight-line interval joining points a and b by $I(a, b)$ and we write

$$X = I(q, q')\cup\bigcup_{n=1}^{\infty}\big(I(p'_n, q_n)\cup I(p_n, q'_n)\big).$$

Obviously, X is a plane smooth dendroid (for the definition of the smoothness see [3], p. 298). We define a mapping f of X onto $f(X)$ similarly as in Example 1:

$$f(r, \varphi) = \begin{cases}(r, 2\varphi) & \text{if } 0\leqslant\varphi\leqslant\pi,\\ \big(r, 2(\varphi-\pi)\big) & \text{if } \pi\leqslant\varphi\leqslant 2\pi.\end{cases}$$

Then f is an open light mapping of X onto the plane smooth dendroid $f(X)$. Since no subcontinuum of X is homeomorphic to $f(X)$, we infer that there exists no subcontinuum L of X such that $f|L$ is a homeomorphism of L onto $f(X)$.

Recall that a continuum X is said to be *arc-like* if for each positive number ε it can be covered by a finite collection of open sets $G_1, G_2, \ldots, G_k$ such that each G_i is of diameter less than ε, and G_i intersects G_j if and only if $|i-j|\leqslant 1$ (see [1], p. 653).

Read asked the following question in [6], p. 54 (P 957):

Is it true that if f is an irreducibly confluent mapping from a hereditarily unicoherent continuum onto an arc-like continuum, then f is irreducible?

The following example gives a negative answer to this problem:

Example 3. Let (x, y) denote a point of the Euclidean plane having x and y as its rectangular coordinates. Put

$$X = \{(x, \sin\pi/x)\colon\ 0< x\leqslant 1\}\cup\{(x, y)\colon\ x = 0 \text{ or } -1 \text{ and } -1\leqslant y\leqslant 1\}\cup$$
$$\cup\{(x, 1)\colon\ -1\leqslant x\leqslant 0\}\cup\big\{\big(x, 1/2+(1/2)\sin\pi/x+1\big)\colon\ -2\leqslant x< -1\big\},$$

and define a mapping f of X onto $f(X)$ by the formula

$$f(x, y) = \begin{cases}(x, y) & \text{if } x\geqslant 0,\\ (0, y) & \text{if } -1\leqslant x\leqslant 0,\\ (x+1, y) & \text{if } x\leqslant -1.\end{cases}$$

It is easily seen that X is a hereditarily unicoherent continuum, $f(X)$ is an arc-like continuum and f is not irreducible, but f is an irreducibly confluent mapping.

Let us consider the following implication: if f is an open light mapping of a compact space X onto a dendrite Y, then there is a subcontinuum L of X such that f restricted to L is a homeomorphism of L onto Y (see [7], Theorem 2.4, p. 188). Whyburn proved this implication firstly in case where Y is an arc (see [7], Theorem 2.1, p. 186). One may suppose that this implication holds also if Y is an arc-like continuum. However, such an assumption is false, which follows from

Example 4. Let (x, y) denote a point of the Euclidean plane having x and y as its rectangular coordinates. Put

$$M = \{(x, \sin \pi/x)\colon\ 0 < x \leqslant 1\} \cup \{(0, y)\colon\ -1 \leqslant y \leqslant 1\},$$
$$N = \{(x, y)\colon\ (-x,\ -y) \in M\}$$

and

$$X = M \cup N.$$

We define an open light mapping f from X onto $f(X)$ by the formula

$$f(x, y) = (|x|, |y|) \quad \text{for each } (x, y) \in X.$$

It is easily seen that X and $f(X)$ are arc-like continua and there exists no subcontinuum L of X such that $f|L$ is a homeomorphism of L onto $f(X)$.

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