COLLOQUIUM MATHEMATICUM

VOL. XXXVII 1977 FASC. 2

A THEOREM OF MAHLER AND SOME APPLICATIONS TO TRANSFERENCE THEOREMS

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Let k be an A-field of characteristic $p \neq 0$ and genus g, and let F_q be its field of constants. Let u be a place of k of degree d, k_u the completion of k at u, and $\mathfrak{o}_u$ the ring of u-exceptional integers of k, i.e., those elements x of k for which $\mathrm{ord}_v(x) \geqslant 0$ for each place $v \neq u$ of k. Let E be the space of n-tuples of elements of k; for any place v of k, let E_v denote the space of n-tuples of elements of k_v. As k_u is locally compact, there is a unique Haar measure in E_u such that the measure of r_u^n is 1; here r_u denotes the maximal compact subring of k_u. In paper [3], the authors proved the following assertion: if $\sigma_1, \ldots, \sigma_n$ are the successive minima of a k_u-lattice ("convex body") L_u of volume V and if $x^{(1)}, \ldots, x^{(n)}$ are independent points of $\mathfrak{o}_u^n$ where these are attained, then

$$V^{-1} \leqslant \sigma_1 \ldots \sigma_n \leqslant V^{-1} q^{n(g+d-1)}, \tag{1}$$

$$1 \leqslant |\det(x_j^{(i)})|_u \leqslant q^{n(g+d-1)}. \tag{2}$$

In the case considered by Mahler, $g = 0$ and $d = 1$, so that

$$\sigma_1 \ldots \sigma_n = V^{-1}, \quad |\det(x_j^{(i)})|_u = 1;$$

as $\det(x_j^{(i)})$ is a non-zero element of $\mathfrak{o}_u$, it is, in fact, an element of F_q. These are precisely the results of Mahler [4]. A natural question (answered automatically in Mahler's case) is the following: in a more general case considered by us in [3], what can be said about an inequality of type (1) if we wish to insist that $\det(x_j^{(i)})$ be a non-zero element of F_q? We can answer this question in the case where $\mathfrak{o}_u$ is a unique factorization domain. In this connection it is worth mentioning that $\mathfrak{o}_u$ is always a Dedekind domain. In fact, we have

THEOREM 1. *Let P be a finite non-empty set of places of k, and let $\mathfrak{o}_P$ be the ring of P-exceptional integers of k, i.e., those elements x of k for which $\mathrm{ord}_v(x) \geqslant 0$ for all places v of k which are not in P. Then $\mathfrak{o}_P$ is a Dedekind domain.*

Proof. Introduce the semigroup $\mathscr{D}_P^+$ of divisors of the kind $\sum_{v \notin P} a(v) \cdot v$, where $a(v)$ are non-negative integers, mostly zero, and consider the homomorphism

$$x \to \mathrm{div}_P(x) = \sum_{v \notin P} \mathrm{ord}_v(x) \cdot v$$

of the multiplicative semigroup of non-zero elements of $\mathfrak{o}_P$ into $\mathscr{D}_P^+$. Let α and β be two non-zero elements of $\mathfrak{o}_P$, and let $\mathfrak{a}$ be an element of $\mathscr{D}_P^+$; it is clear that

(i) α divides β if and only if $\mathrm{div}_P(\alpha) \leqslant \mathrm{div}_P(\beta)$;

(ii) if $\mathrm{div}_P(\alpha)$ and $\mathrm{div}_P(\beta)$ are both not less than $\mathfrak{a}$, then $\mathrm{div}_P(\alpha \pm \beta) \geqslant \mathfrak{a}$ ($\mathrm{div}_P(0)$ is supposed to be not less than each element of $\mathscr{D}_P^+$).

Suppose next that

$$\mathfrak{a} = \sum_{v \notin P} a(v) \cdot v \quad \text{and} \quad \mathfrak{b} = \sum_{v \notin P} b(v) \cdot v$$

are two divisors in $\mathscr{D}_P^+$ such that $\mathrm{div}_P(x) \geqslant \mathfrak{a}$ whenever $\mathrm{div}_P(x) \geqslant \mathfrak{b}$; we claim that $\mathfrak{b} \geqslant \mathfrak{a}$. For, otherwise, there exists a place $v \notin P$ such that $b(v) < a(v)$. Let

$$\mathfrak{b}' = \sum_{u \in P} b(u) \cdot u - \sum_{w \notin P} b(w) \cdot w$$

be so chosen that $\deg(\mathfrak{b}') > 2g - 2 + \deg(v)$. Then, by the Riemann-Roch theorem,

$$\lambda(\mathfrak{b}') = \deg(\mathfrak{b}') - g + 1$$

and

$$\lambda(\mathfrak{b}' - v) = \deg(\mathfrak{b}') - g + 1 - \deg(v).$$

Consequently, there exists α in $\Lambda(\mathfrak{b}')$ which is not in $\Lambda(\mathfrak{b}' - v)$; this α is in $\mathfrak{o}_P$, $\mathrm{div}_P(\alpha) \geqslant \mathfrak{b}$ and $\mathrm{ord}_v(\alpha) = b(v) < a(v)$, so that $\mathrm{div}_P(\alpha) \not\geqslant \mathfrak{a}$, giving us the desired contradiction.

Thus, we have introduced a theory of divisors in $\mathfrak{o}_P$ in the sense of [1]. For $v \notin P$, denote by $\mathfrak{p}_v$ the set of those elements x in $\mathfrak{o}_P$ for which $\mathrm{ord}_v(x) \geqslant 1$; $\mathfrak{p}_v$ is obviously a prime ideal in $\mathfrak{o}_P$. If r_v denotes the maximal compact subring of k_v, and p_v is the prime ideal in r_v, then the map

$$(*) \qquad \mathfrak{o}_P/\mathfrak{p}_v \to r_v/p_v$$

is well defined and injective; as r_v/p_v is a finite field, the same is true for $\mathfrak{o}_P/\mathfrak{p}_v$. It follows that $\mathfrak{o}_P$ is a Dedekind domain.

Remark 1. The map $(*)$ is surjective. In fact, let $x \in r_v$, and choose a divisor

$$\mathfrak{a} = \Big(\sum_{u \in P} a(u) \cdot u\Big) - v$$

of degree greater than $2g-2$. Then, by [5],

$$k_A = k + \prod_{u \in P} p_u^{-a(u)} \times p_v \times \prod_{v \neq w \notin P} r_w,$$

and hence the adele with the v-th component x and the remaining components 0 can be expressed as $y+z$, where y is in k, and z is an element of

$$\prod_{u \in P} p_u^{-a(u)} \times p_v \times \prod_{v \neq w \notin P} r_w.$$

For each $w \neq v$, $w \notin P$, we have $y+z_w = 0$, and hence $y = -z_w \in r_w$. Moreover, $x-y = z_v$ is in p_v; as x is in r_v, y is also in r_v. Thus y is in $\mathfrak{o}_P$, and $x-y$ is in p_v.

Remark 2. If x is any non-zero element of k, we can, by the Riemann-Roch theorem, find an element $\beta \neq 0$ of $\mathfrak{o}_P$ such that, for each $v \notin P$,

$$\mathrm{ord}_v(\beta) \geqslant -\mathrm{ord}_v(x).$$

Then βx is an element of $\mathfrak{o}_P$, and k is the field of quotients of $\mathfrak{o}_P$.

Remark 3. The mapping of $k_A^\times$ (idele group of k) into the group of ideals of $\mathfrak{o}_P$ defined by

$$z \to \prod_{v \notin P} \mathfrak{p}_v^{\mathrm{ord}_v(z_v)}$$

is surjective with the kernel

$$\Omega(P) = \prod_{u \in P} k_u^\times \times \prod_{v \notin P} r_v^\times.$$

Thus the ideal class group of $\mathfrak{o}_P$ is isomorphic to $k_A^\times / k^\times \Omega(P)$ and is therefore finite [5], of order h_P. The ring $\mathfrak{o}_P$ is a unique factorization domain if and only if $h_P = 1$, in which case $\mathfrak{o}_P$ is a principal ideal domain.

We now prove two lemmas which we shall need in the sequel.

LEMMA 1. *If the field k and the place u of k are such that $\mathfrak{o}_u$ is a unique factorization domain, then the degree d of the place must be* 1, *so that $q_u = q$.*

Proof. It is well known (see [5], Corollary 5, Theorem 2, Section 6, Chapter VII) that we can find a divisor

$$\mathfrak{a} = \sum_v a(v) \cdot v$$

of degree 1 with $a(u) = 0$. As $\mathfrak{o}_u$ is a unique factorization domain, there exists x in k such that $\mathrm{ord}_v(x) = a(v)$ for all $v \neq u$. Now

$$\begin{aligned} 0 &= \mathrm{ord}_u(x)\deg(u) + \sum_{v \neq u} \mathrm{ord}_v(x)\deg(v) \\ &= \mathrm{ord}_u(x)\deg(u) + \sum_{v \neq u} a(v)\deg(v) = \mathrm{ord}_u(x)\deg(u) + 1 \end{aligned}$$

and it follows that $\deg(u) = 1$.

LEMMA 2. *Let k be an A-field, and u a place of k of degree d (not necessarily such that $\mathfrak{o}_u$ is a unique factorization domain). Then there exists an integer a such that*

(i) *for any a in k_u,*

$$\|a\|_u = \operatorname*{Min}_{x \in \mathfrak{o}} |x-a|_u \leqslant q_u^a;$$

(ii) *there exists β in k_u with $\|\beta\|_u = q_u^a$.*

This integer a satisfies the inequalities

$$g-1 \leqslant ad \leqslant 2g-2+d. \tag{3}$$

In particular, for any a in k_u, the inequality

$$|x-a|_u \leqslant q^{2g-2+d} \tag{4}$$

can always be solved for an x in $\mathfrak{o}_u$.

Proof. Let us first notice that condition (i) is equivalent to

$$k_A = k + p_u^{-a} \times \prod_{v \neq u} r_v. \tag{5}$$

To show this, suppose that (5) is satisfied. Denote by a ($a \in k_u$) again the adele for which $a_u = a$, and $a_v = 0$ for $v \neq u$. By (5), we can write $a = x+z$ with x in k and z in $p_u^{-a} \times \prod_{v \neq u} r_v$. Then $x = -z_v \in r_v$ for all $v \neq u$, and hence x is in $\mathfrak{o}_u$. Moreover,

$$|x-a|_u = |z_u|_u \leqslant q_u^a.$$

Conversely, suppose that the integer a satisfies condition (i) of the lemma. Regarding k_u as a part of the adele ring k_A, we then have

$$k_u \subset k + p_u^{-a} \times \prod_{v \neq u} r_v,$$

and hence

$$k + k_u \subset k + p_u^{-a} \times \prod_{v \neq u} r_v. \tag{6}$$

Since $k+k_u$ is dense in k_A and since the right-hand side of (6) is an open, and hence closed, subgroup of k_A, formula (5) is true.

Suppose now that (5) holds for an integer a. Let $L = (L_v)_v$ be the coherent system of lattices (belonging to the vector space k) defined as follows: $L_u = p_u^{-a}$, and $L_v = r_v$ for $v \neq u$. Then, by (5), $\lambda(L') = 0$ (see [5]), and hence

$$-\delta(L) = \lambda(L) + g - 1 \geqslant g-1,$$

so that

$$q_u^a = \operatorname{meas}\Bigl(p_u^{-a} \times \prod_{v \neq u} r_v\Bigr) = q^{-\delta(L)} \geqslant q^{g-1}$$

i.e., $ad \geqslant g-1$.

Now consider the divisor au, a being any integer such that $ad > 2g-2$. Then (see [5])

$$k_A = k + p_u^{-a} \times \prod_{v \neq u} r_v .$$

All our assertions are now obvious if we take a to be the least integer for which (5) holds.

Coming back to the question asked at the beginning, we prove

THEOREM 2. *Let u be a place of k such that $\mathfrak{o}_u$ is a unique factorization domain. Let L_u be a k_u-lattice ("convex body") in E_u of volume V (with respect to the Haar measure of E_u for which the measure of r_u^n is* 1) *and let*

$$F_u(x) = \operatorname*{Inf}_{\substack{a \neq 0 \\ ax \in L_u}} |a|_u^{-1}$$

be the associated norm function. Then there exist n points $y^{(1)}, \ldots, y^{(n)}$ in $\mathfrak{o}_u^n$ such that

$$|\det(y_j^{(i)})|_u = 1 \tag{7}$$

and

$$V^{-1} \leqslant \prod_{i=1}^{n} F_u(y^{(i)}) \leqslant V^{-1} q^{3ng}. \tag{8}$$

Proof. Let $\sigma_1, \ldots, \sigma_n$ be the successive minima of L_u and let $z^{(1)}, \ldots$ $\ldots, z^{(n)}$ be n independent (over k_u) points of $\mathfrak{o}_u^n$, where the minima are attained. Let Z be the matrix whose columns are $z^{(1)}, \ldots, z^{(n)}$. Since $\mathfrak{o}_u$ is a principal ideal domain, we can find a matrix A with entries from $\mathfrak{o}_u$ such that $\det(A)$ is a unit of $\mathfrak{o}_u$, i.e., a non-zero element of F_q, and

$$AZ = \begin{pmatrix} x_1^{(1)} & \ldots & & x_1^{(n)} \\ 0 & x_2^{(2)} & \ldots & x_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \ldots & 0 & x_n^{(n)} \end{pmatrix}$$

has zero entries below the main diagonal. Let

$$x^{(i)} = (x_1^{(i)}, \ldots, x_i^{(i)}, 0, \ldots, 0) \quad (1 \leqslant i \leqslant n). \tag{9}$$

Define a new distance function $G_u(x)$ by

$$G_u(x) = F_u(A^{-1}x). \tag{10}$$

Since $\det(A)$ is a non-zero member of F_q, the function G_u has the same successive minima as F_u and they are attained at $x^{(1)}, \ldots, x^{(n)}$, respectively. We notice that $x_1^{(1)}$ is a unit and write

$$x'^{(1)} = (x_1^{(1)})^{-1}x^{(1)} = (1, 0, \ldots, 0).$$

Then

$$G_u(x'^{(1)}) = \sigma_1.$$

Similarly, if $x_i^{(i)}$ is a unit of $\mathfrak{o}_u$ for some $i \geqslant 2$, we set

$$x'^{(i)} = (x_i^{(i)})^{-1}x^{(i)},$$

so that

$$G_u(x'^{(i)}) = \sigma_i.$$

Suppose now that $i \geqslant 2$ is such that $x_i^{(i)}$ is not a unit, and hence $|x_i^{(i)}|_u \geqslant q_u$. Now write

$$(0, \ldots, 0, 1, 0, \ldots, 0) = \beta_1 x^{(1)} + \ldots + \beta_{i-1}x^{(i-1)} + (x_i^{(i)})^{-1}x^{(i)}$$

(1 being at the i-th place).

Find, by Lemma 2, elements $b_1, \ldots, b_{i-1}$ of $\mathfrak{o}_u$ such that

$$|b_j - \beta_j| \leqslant q^{2g}$$

and set

$$x'^{(i)} = (0, \ldots, 0, 1, 0, \ldots, 0) - \sum_{j=1}^{i-1} b_j x^{(j)}.$$

Then $x'^{(i)}$ has 1 at its i-th place and zero afterwards; moreover,

$$G_u(x'^{(i)}) = G_u\Big(\sum_{j=1}^{i-1} (\beta_j - b_j)x^{(j)} + (x_i^{(i)})^{-1}x^{(i)}\Big) \leqslant q^{2g}\sigma_i.$$

Thus we have found points $x'^{(i)}$ $(1 \leqslant i \leqslant n)$ in $\mathfrak{o}_u^n$ such that $x'^{(i)}$ has 1 at its i-th place and zero afterwards, so that

$$\det(x_j'^{(i)}) = 1. \tag{11}$$

The inequality $G_u(x'^{(i)}) \leqslant q^{2g}\sigma_i$, by (1) and the fact that $d = 1$, implies

$$\prod_{i=1}^{n} G_u(x'^{(i)}) \leqslant q^{2ng}\sigma_1 \ldots \sigma_n \leqslant q^{3ng}V^{-1}. \tag{12}$$

Notice that the volume of $G_u(x) \leqslant 1$ is the same as that of $F_u(x) \leqslant 1$. Define $y^{(1)}, \ldots, y^{(n)}$ by $Ay^{(i)} = x'^{(i)}$. Since $\det(A)$ is a non-zero member of F_q, $y^{(i)}$ are in $\mathfrak{o}_u^n$, and $\det(y_j^{(i)})$ is a non-zero member of F_q, which proves (7). Moreover, by (10) and (12) we have

$$\prod_{i=1}^{n} F_u(y^{(i)}) = \prod_{i=1}^{n} G_u(x'^{(i)}) \leqslant q^{3ng} V^{-1},$$

which is the second of inequalities (8). To prove the first one, without loss of generality suppose that

$$F_u(y^{(1)}) \leqslant \ldots \leqslant F_u(y^{(n)}).$$

Then the k_u-lattice $F_u(x) \leqslant F_u(y^{(i)})$ contains the i independent points $y^{(1)}, \ldots, y^{(i)}$ of $\mathfrak{o}_u^n$, and hence

$$\sigma_i \leqslant F_u(y^{(i)}).$$

Consequently, by (1) we have

$$V^{-1} \leqslant \sigma_1 \ldots \sigma_n \leqslant \prod_{i=1}^{n} F_u(y^{(i)}).$$

This completes the proof of Theorem 2.

We now give some applications of Theorem 2. In the sequel, k is an A-field of characteristic $p \neq 0$, and u is a place of k such that $\mathfrak{o}_u$ is a unique factorization domain; by Lemma 1, the degree of u is 1.

Suppose that

$$(13) \qquad \varphi_\lambda(z) = \sum_{\mu=1}^{l} \varphi_{\lambda\mu} z_\mu \quad \text{and} \quad \psi_\lambda(w) = \sum_{\mu=1}^{l} \psi_{\lambda\mu} w_\mu \qquad (1 \leqslant \lambda \leqslant l)$$

are linear forms over k_u such that we have identically

$$(14) \qquad \sum_\lambda \varphi_\lambda(z)\psi_\lambda(w) = \sum_\lambda z_\lambda w_\lambda.$$

Let $\beta = (\beta_1, \ldots, \beta_l)$ be an arbitrary (but fixed) element of k_u^l. Suppose that there exists $b = (b_1, \ldots, b_l)$ in $\mathfrak{o}_u^l$ such that

$$(15) \qquad |\beta_\lambda - \varphi_\lambda(b)|_u \leqslant 1 \qquad (1 \leqslant \lambda \leqslant l).$$

Then, for every w in $\mathfrak{o}_u^l$,

$$\sum_\lambda \varphi_\lambda(b)\psi_\lambda(w) = \sum_\lambda b_\lambda w_\lambda$$

is an element of $\mathfrak{o}_u$, and hence

$$(16) \qquad \Big\| \sum_\lambda \beta_\lambda \psi_\lambda(w) \Big\|_u \leqslant \Big| \sum_\lambda \big(\beta_\lambda - \varphi_\lambda(b)\big)\psi_\lambda(w) \Big|_u \leqslant \max_\lambda |\psi_\lambda(w)|_u.$$

Conversely, suppose that for every w in $\mathfrak{o}_u^l$

$$(17) \qquad \Big\| \sum_\lambda \beta_\lambda \psi_\lambda(w) \Big\|_u \leqslant q^{-3lg} \max_\lambda |\psi_\lambda(w)|_u.$$

Then, we claim that there exists b in $\mathfrak{o}_u^l$ for which (15) holds. To prove this, let $\Phi = (\varphi_{\lambda\mu})$ and $\Psi = (\psi_{\lambda\mu})$ be the matrices of the linear forms φ_λ $(1 \leqslant \lambda \leqslant l)$ and ψ_λ $(1 \leqslant \lambda \leqslant l)$, respectively. Then, by (14), we have

(14′) $$\Psi' = \Phi^{-1}.$$

In what follows, w will denote a row vector while z and β will stand for column vectors. Applying Theorem 2 to the lattice

$$\max_j |\psi_j(w)|_u \leqslant 1$$

find an $(l \times l)$-matrix

(18) $$W = \begin{pmatrix} w^{(1)} \\ \cdots \\ w^{(l)} \end{pmatrix}$$

with entries in $\mathfrak{o}_u$ such that

(19) $$\det(w_j^{(i)}) = 1, \qquad \sigma_1 \ldots \sigma_l \leqslant q^{3lg} |\det \Psi|_u,$$

where

(19′) $$\sigma_\lambda = \max_j |\psi_j(w^{(\lambda)})|_u \qquad (1 \leqslant \lambda \leqslant l).$$

Now $W\Psi'\beta$ is a column vector with the λ-th entry $\sum_j \beta_j \psi_j(w^{(\lambda)})$. Hence, by (17) and (19′), we can write

(20) $$W\Psi'\beta = a + \delta,$$

where a is in $\mathfrak{o}_u^l$, and the vector δ satisfies

(21) $$|\delta_\lambda|_u \leqslant q^{-3lg} \sigma_\lambda.$$

By (20) and (14′), we have

(22) $$\beta = \Phi b + \gamma,$$

where

(23) $$b = W^{-1}a \qquad \text{and} \qquad \delta = W\Psi'\gamma.$$

As W is unimodular and a is in $\mathfrak{o}_u^l$, we see that b is also in $\mathfrak{o}_u^l$. Solving the equations $W\Psi'\gamma = \delta$ by Cramer's rule, we see that

(24) $$\gamma_j = (\det W\Psi')^{-1} \det(\Theta_j) = (\det \Psi)^{-1} \det(\Theta_j),$$

where Θ_j is the matrix obtained from $W\Psi'$ by replacing its j-th column by δ; the λ-th row in $W\Psi'$ is $(\psi_1(w^{(\lambda)}), \ldots, \psi_l(w^{(\lambda)}))$, and each of these quantities is absolutely not greater than σ_λ. Using (19) and (21), we easily get

$$|\det(\Theta_j)|_u \leqslant q^{-3lg} q^{3lg} |\det \Psi|_u,$$

so that $|\gamma_j|_u \leqslant 1$. This, along with (22), proves pur assertion.

Thus we have

THEOREM 3. *Let φ_λ and ψ_λ $(1 \leqslant \lambda \leqslant l)$ be linear forms over k_u such that* (14) *holds. Let $\beta = (\beta_1, \ldots, \beta_l)$ be in k_u^l.*

(A) *If there exists b in $\mathfrak{o}_u^l$ such that* (15) *holds, then* (16) *is true for every w in $\mathfrak{o}_u^l$.*

(B) *If* (17) *holds for every w in $\mathfrak{o}_u^l$, then there exists b in $\mathfrak{o}_u^l$ for which* (15) *holds.*

Let now

$$L_j(x) = \sum_{i=1}^{m} \theta_{ji} x_i \quad (1 \leqslant j \leqslant n) \tag{25}$$

be n linear forms over k_u in m variables $x_1, \ldots, x_m$, and let

$$M_i(y) = \sum_{j=1}^{n} \theta_{ji} y_j \quad (1 \leqslant i \leqslant m) \tag{25'}$$

be the transposed system. Let $a = (a_1, \ldots, a_n)$ be any vector k_u^n, and let s, t be integers not less than 0. Let $l = m+n$ and let π be a prime element of k_u. Applying Theorem 3 to the forms

$$\varphi_\lambda(z) = \varphi_\lambda(x_1, \ldots, x_m, y_1, \ldots, y_n) = \begin{cases} \pi^{-t}\big(L_\lambda(x) + y_\lambda\big) & (\lambda \leqslant n), \\ \pi^s x_{\lambda-n} & (\lambda > n), \end{cases}$$

$$\psi_\lambda(w) = \psi_\lambda(u_1, \ldots, u_m, v_1, \ldots, v_n) = \begin{cases} \pi^t v_\lambda & (\lambda \leqslant n), \\ \pi^{-s}\big(u_{\lambda-n} - M_{\lambda-n}(v)\big) & (\lambda > n), \end{cases}$$

and to the vector

$$\beta = (\beta_1, \ldots, \beta_l) = \pi^{-t}(a_1, \ldots, a_n, 0, \ldots, 0),$$

we obtain

THEOREM 4. *Let L_j $(1 \leqslant j \leqslant n)$ and M_i $(1 \leqslant i \leqslant m)$ be the linear forms* (25) *and* (25′), *respectively. Let $a = (a_1, \ldots, a_n)$ be any vector in k_u^n and let $s, t \geqslant 0$.*

(A) *If there exists a in $\mathfrak{o}_u^m$ such that*

$$\|L_j(a) - a_j\|_u \leqslant q^{-t}, \quad |a_i|_u \leqslant q^s \quad (1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n), \tag{26}$$

then, for all c in $\mathfrak{o}_u^n$,

$$\Big\|\sum_{j=1}^{n} c_j a_j\Big\|_u \leqslant \max_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}} \big(q^s \|M_i(c)\|_u, q^{-t} |c_j|_u\big). \tag{27}$$

(B) *If, for every* c *in* $\mathfrak{o}_u^n$,

$$(27') \qquad \Big\|\sum_{j=1}^{n} c_j a_j\Big\|_u \leqslant q^{-3(m+n)g} \max_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} \big(q^s\|M_i(c)\|_u,\, q^{-t}|c_j|_u\big),$$

then (26) *can be solved for an* a *in* $\mathfrak{o}_u^m$.

Finally, noting that, for any $c = (c_1, \ldots, c_n)$ in $\mathfrak{o}_u^n$,

$$\sum_{j=1}^{n} c_j L_j(x) = \sum_{i=1}^{m} M_i(c) x_i,$$

we easily obtain, by Theorem 4, the following

THEOREM 5 (Kronecker). *Let* $L_j(x_1, \ldots, x_m)$ $(1 \leqslant j \leqslant n)$ *be* n *linear forms in* m *variables, and let* $a = (a_1, \ldots, a_n)$ *be any vector in* k_u^n. *Then the following two statements are equivalent*:

(A) *For each* $t > 0$, *there exists* a *in* $\mathfrak{o}_u^m$ *such that*

$$(28) \qquad \|L_j(a) - a_j\|_u \leqslant q^{-t} \qquad (1 \leqslant j \leqslant n).$$

(B) *If* $c = (c_1, \ldots, c_n)$ *is any vector in* $\mathfrak{o}_u^n$ *such that the form* $c_1L_1 + \ldots + c_nL_n$ *has coefficients in* $\mathfrak{o}_u$, *then* $c_1a_1 + \ldots + c_na_n$ *is an element of* $\mathfrak{o}_u$.

Proof. Suppose that (A) holds, and let c in $\mathfrak{o}_u^n$ be such that $c_1L_1 + \ldots + c_nL_n$ has coefficients in $\mathfrak{o}_u$, i.e., $\|M_i(c)\|_u = 0$ for $1 \leqslant i \leqslant m$. Let $t > 0$ be arbitrary and let a (in $\mathfrak{o}_u^m$) satisfy (28). Then, putting

$$q^s = \max_i |a_i|_u,$$

by Theorem 4 we have

$$\Big\|\sum_{j=1}^{n} c_j a_j\Big\|_u \leqslant \max_{i,j} \big(q^s\|M_i(c)\|_u,\, q^{-t}|c_j|_u\big) = q^{-t}\max_j |c_j|_u.$$

Letting t approach ∞, we see that

$$\Big\|\sum_j c_j a_j\Big\|_u = 0,$$

i.e., $\sum_j c_j a_j$ is in $\mathfrak{o}_u$.

Suppose now that (B) holds. Let $t > 0$ be any integer. Inequality (27′) is satisfied by every c in $\mathfrak{o}_u^n$ except perhaps finitely many c for which

$$(29) \qquad \max_j |c_j|_u < q^{t+2g-1+3(m+n)g}$$

(notice that $\|\sum_j c_j a_j\|_u \leqslant q^{2g-1}$ by Lemma 2).

If an element c of $\mathfrak{o}_u^n$ satisfying (29) is such that $\|M_i(c)\|_u = 0$ for $1 \leqslant i \leqslant m$, then

$$\Big\|\sum_j c_j a_j\Big\|_u = 0,$$

and hence c satisfies (27′). For the remaining finitely many c in $\mathfrak{o}_u^n$ satisfying (29), we can obviously choose s such that (27′) is satisfied. It follows from Theorem 4 that the inequalities

$$\|L_j(a) - a_j\|_u \leqslant q^{-t} \quad \text{and} \quad |a_i|_u \leqslant q^s$$

can be solved for a in $\mathfrak{o}_u^m$. This completes the proof of Theorem 5.

COROLLARY (Kronecker). *Let $\theta_1, \ldots, \theta_n$ be n elements of k_u such that $1, \theta_1, \ldots, \theta_n$ are linearly independent over k. Then, for any $\alpha = (\alpha_1, \ldots, \alpha_n)$ in k_u^n and for any integer $t > 0$, we can find an element a of $\mathfrak{o}_u$ such that*

$$\|\theta_j a - \alpha_j\|_u \leqslant q^{-t}.$$

Proof. Take $L_j(x) = \theta_j x$ $(1 \leqslant j \leqslant n)$ and notice that condition (B) of Theorem 5 is automatically satisfied.

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Reçu par la Rédaction le 25. 2. 1976