COLLOQUIUM MATHEMATICUM

VOL. XXXIII 1975 FASC. 2

GENERALIZATIONS OF ASCOLI'S THEOREMS

BY

PAWEŁ ROLICZ (SZCZECIN)

This paper contains the generalizations of the Ascoli theorems.

In [4] A. K. Steiner and E. F. Steiner have introduced the concept of semi-uniformities. A *semi-uniform space* consists of a pair (Y, C), where Y is a set and C is a family of coverings of Y, satisfying the following conditions:

(i) If $\tau \in C$, then there exists $\tau' \in C$ such that, for every $T' \in \tau'$, there are $T \in \tau$ and $\tau'' \in C$ with the property $\mathrm{st}(T', \tau'') \subset T$.

(ii) If $\tau, \tau' \in C$, then there exists $\tau'' \in C$ such that τ'' refines both τ and τ'.

(iii) If $\tau' \in C$ refines a covering τ, then $\tau \in C$.

(iv) For distinct points $x, y \in Y$, there exists $\tau \in C$ such that $y \notin \mathrm{st}(x, \tau)$.

The family C is called a *semi-uniformity* on Y. Members of C are said to be *semi-uniform coverings*.

Each semi-uniformity C on Y determines a topology $\tau(C)$ on Y as follows: $U \in \tau(C)$ iff, for each $y \in U$, there is a $\tau \in C$ such that $\mathrm{st}(y, \tau) \subset U$.

For every semi-uniformity C, $(Y, \tau(C))$ is a T_3-space. Each semi-uniform covering in C has a semi-uniform refinement open in $\tau(C)$.

Throughout, C denotes a semi-uniformity on Y and the space $(Y, \tau(C))$ is denoted by Y. Moreover, we assume the following notation:

$$\mathrm{st}^{(1)}(y, \tau) = \mathrm{st}(y, \tau), \quad \mathrm{st}^{(n+1)}(y, \tau) = \mathrm{st}\big(\mathrm{st}^{(n)}(y, \tau), \tau\big).$$

If $y \in Y$ and $\tau \in C$, then there exists $\tau' \in C$ such that

$$\mathrm{st}^{(2)}(y, \tau') \subset \mathrm{st}(y, \tau).$$

Consequently, for each $y \in Y$, $\tau \in C$ and positive integers n, there exists $\tau' \in C$ such that

$$\mathrm{st}^{(n)}(y, \tau') \subset \mathrm{st}(y, \tau).$$

LEMMA 1. *Let Z be a compact subset of a space Y.*

(a) *If $U_1, \ldots, U_k$ are open in Y and $Z \subset U_1 \cup \ldots \cup U_k$, then there exists $\tau \in C$ such that*

$$\mathrm{st}^{(n)}(y, \tau) \subset \bigcup\{U_i:\ y \in U_i\} \quad \textit{for every } y \in Z.$$

(b) *For every $\tau \in C$ there exists $\tau' \in C$ such that*

$$\mathrm{st}^{(n)}(y, \tau') \subset \mathrm{st}(y, \tau) \quad \textit{for every } y \in Z.$$

Proof. For every $y \in Z$ there exists $\tau_y \in C$ such that

$$\mathrm{st}^{(n+1)}(y, \tau_y) \subset \bigcap\{U_i:\ y \in U_i\}.$$

Since Z is compact,

$$Z \subset \bigcup\{\mathrm{st}(y_j, \tau_{y_j}):\ 1 \leqslant j \leqslant l\}.$$

There is $\tau \in C$ which refines τ_{y_j} for each j. Let $y \in Z$. There is $j \leqslant l$ such that $y \in \mathrm{st}(y_j, \tau_{y_j})$. Thus

$$\mathrm{st}(y, \tau) \subset \mathrm{st}^{(2)}(y_j, \tau_{y_j})$$

and, consequently,

$$\mathrm{st}^{(n)}(y, \tau) \subset \mathrm{st}^{(n+1)}(y_j, \tau_{y_j}) \subset \bigcup\{U_i:\ y \in U_i\}.$$

Since each $\tau \in C$ has an open semi-uniform refinement, (b) follows directly from (a).

Let X be a topological space. The set of all continuous functions from X to Y will be denoted by Y^X. Let K be a family of subsets of X directed with respect to inclusion (i.e., for every $Z_1, Z_2 \in K$, there exists $Z_3 \in K$ such that $Z_1 \cup Z_2 \subset Z_3$). We introduce the following symbols:

$$B_Z(f, \tau) = \{g \in Y^X:\ g(x) \in \mathrm{st}(f(x), \tau) \text{ for every } x \in Z\},$$

$$\hat{\tau}|Z = \{B_Z(f, \tau):\ f \in Y^X\}, \qquad \hat{C}|K = \{\hat{\tau}|Z:\ \tau \in C, Z \in K\}.$$

The family $\hat{C}|K$ determines a topology $\tau(\hat{C}|K)$ on Y^X as follows: $F \in \tau(\hat{C}|K)$ iff, for each $f \in F$, there is a $\hat{\tau}|Z \in \hat{C}|K$ such that $\mathrm{st}(f, \hat{\tau}|Z) \subset F$.

Throughout, K denotes the collection of all compact subsets of X, and the space X is assumed to be T_2.

The space $(Y^X, \tau(\hat{C}|K))$ is a T_3-space (see Theorem 1 below).

Example. Let X denote the real numbers with the usual topology, let Y be the real numbers, and let $U_i(n)$ be the open interval of length $1-2^{-n}$ with the center i. Let

$$\tau_i = \bigcup_{j=i}^{\infty}\{U_j(n):\ n = 1, 2, \ldots\} \cup \{\text{all open intervals of length } 2^{-i}\}.$$

The family $\{\tau_i\}$ is a base for a semi-uniformity C on Y. We say that τ' *locally star-refines* τ in C if condition (i) is satisfied. Suppose that $\hat{\tau}_k|Z_k$

locally star-refines $\hat{\tau}_m|Z_m$ in $\hat{C}|K$, where $Z_m = \{m\}$. Let

$$\hat{T}_k = B_{Z_k}(f_1, \tau_k) \in \hat{\tau}_k|Z_k, \quad \text{where } f_1 \in Y^X,\ f_1(x) = k \text{ for every } x \in X.$$

Observe that

$$(*) \quad g \in \mathrm{st}(\hat{T}_k, \hat{\tau}_l|Z_l)$$

$$\Leftrightarrow \bigvee_{h, h_1 \in Y^X} \bigwedge_{x_k \in Z_k} h_1(x_k) \in \mathrm{st}\big(f_1(x_k), \tau_k\big) \text{ and } \bigwedge_{x_l \in Z_l} h_1(x_l), g(x_l) \in \mathrm{st}\big(h(x_l), \tau_l\big).$$

If $f \in Y^X$ and $f(m) \leqslant k$, then $g_1 \notin B_{Z_m}(f, \tau_m)$, where $g_1(x) = k + \frac{1}{2}$ for every $x \in X$. Let h, $h_1 \in Y^X$, $h(x) = h_1(x) = k + \frac{1}{2} - 2^{-l-2}$ for each $x \in X$. It follows from $(*)$ that $g_1 \in \mathrm{st}(\hat{T}_k, \hat{\tau}_l|Z_l)$. Similarly, if $f_1(m) \geqslant k$, then

$$g_2 \notin B_{Z_m}(f, \tau_m) \quad \text{and} \quad g_2 \in \mathrm{st}(\hat{T}_k, \hat{\tau}_l|Z_l),$$
$$\text{where } g_2(x) = k - \tfrac{1}{2} \text{ for every } x \in X.$$

This is a contradiction with the assumption. Hence $\hat{C}|K$ is not a base for any semi-uniformity on Y^X.

THEOREM 1. *The natural topology on Y^X and $\tau(\hat{C}|K)$ are identical.*

Proof. Let D denote the natural topology on Y^X. Let $Z \in K$, let U be any open set in Y and

$$f \in P(Z, U), \quad \text{where } P(Z, U) = \{f \in Y^X\colon f[Z] \subset U\}.$$

By Lemma 1, there exists $\tau \in C$ such that $\mathrm{st}^{(2)}\big(f(x), \tau\big) \subset U$ for every $x \in Z$. Thus

$$\mathrm{st}(f, \hat{\tau}|Z) \subset P(Z, U) \quad \text{and} \quad P(Z, U) \in \tau(\hat{C}|K)$$

which proves that $D \subset \tau(\hat{C}|K)$.

We need to show that $\tau(\hat{C}|K) \subset D$. Let $Z \in K$, $\tau \in C$ and $f \in Y^X$. By Lemma 1, there exists τ' such that

$$\mathrm{st}^{(5)}\big(f(x), \tau'\big) \subset \mathrm{st}\big(f(x), \tau\big) \quad \text{for every } x \in Z.$$

Since Z is compact,

$$f[Z] \subset \bigcup \big\{\mathrm{st}\big(f(x_i), \tau'\big)\colon 1 \leqslant i \leqslant k\big\}.$$

Let

$$Z_i = Z \cap f^{-1}\big[\mathrm{cl}\,\mathrm{st}\big(f(x_i), \tau'\big)\big], \quad U_i = \mathrm{Int}\,\mathrm{st}^{(3)}\big(f(x_i), \tau'\big).$$

Since $\mathrm{cl}\,\mathrm{st}\big(f(x_i), \tau'\big) \subset \mathrm{st}^{(2)}\big(f(x_i), \tau'\big)$,

$$\mathrm{st}(y, \tau') \subset \mathrm{st}^{(3)}\big(f(x_i), \tau'\big) \quad \text{for every } y \in \mathrm{cl}\,\mathrm{st}\big(f(x_i), \tau'\big).$$

Thus

$$\mathrm{cl}\,\mathrm{st}\big(f(x_i), \tau'\big) \subset \mathrm{Int}\,\mathrm{st}^{(3)}\big(f(x_i), \tau'\big) \quad \text{and} \quad f \in P(Z_i, U_i)$$
$$\text{for } i = 1, 2, \ldots, k.$$

Let $g \in P(Z_i, U_i)$ for $i = 1, 2, \ldots, k$. For every $x \in Z$ there exists $i \leqslant k$ such that $x \in Z_i$ and, therefore,

$$g(x) \in \mathrm{st}^{(3)}\big(f(x_i), \tau'\big) \quad \text{and} \quad f(x) \in \mathrm{st}^{(2)}\big(f(x_i), \tau'\big).$$

Thus $g(x) \in \mathrm{st}^{(5)}\big(f(x), \tau'\big) \subset \mathrm{st}\big(f(x), \tau\big)$ for $x \in Z$, in other words, $g \in B_Z(f, \tau)$. Hence

$$(1) \qquad f \in \bigcap\{P(Z_i, U_i)\colon 1 \leqslant i \leqslant k\} \subset B_Z(f, \tau) \subset \mathrm{st}(f, \hat{\tau}|Z)$$

which proves that $\tau(\hat{C}|K) \subset D$.

By Theorem 1 and (1) we have the following

COROLLARY. *A family $F \subset Y^X$ is open in the natural topology on Y^X iff, for every $f \in F$, there exist $Z \in K$ and $\tau \in C$ such that $B_Z(f, \tau) \subset F$.*

Functions of a family $F \subset Y^X$ will be called *equally continuous* if, for each $x \in X$ and $\tau \in C$, there exists an open neighborhood $G \ni x$ such that $f[G] \subset \mathrm{st}\big(f(x), \tau\big)$ for every $f \in F$.

LEMMA 2. *Let X be a locally compact space. If a collection $F \subset Y^X$ fs compact in the space Y^X with the natural topology, then functions of the iamily F are equally continuous.*

Proof. Let $x \in X$ and $\tau \in C$. For every $f \in F$ there exist $\tau_f \in C$ and $T_f \in \tau$ such that $\mathrm{st}\big(f(x), \tau_f\big) \subset T_f$. Since X is locally compact, for every $f \in F$ there exists an open neighborhood $G_f \ni x$ such that $\mathrm{cl}G_f$ is compact and $f[\mathrm{cl}G_f] \subset U_f$, where $U_f = \mathrm{Int}\,\mathrm{st}\big(f(x), \tau_f\big)$. The family F is compact. Thus

$$F \subset \bigcup\{P(\mathrm{cl}G_{f_i}, U_{f_i})\colon 1 \leqslant i \leqslant k\}.$$

Let $G = \bigcap\{G_{f_i}\colon 1 \leqslant i \leqslant k\}$. For every $f \in F$ there exists $i \leqslant k$ such that $f \in P(\mathrm{cl}G_{f_i}, U_{f_i})$. If $x' \in G$, then we have $f(x'), f(x) \in U_{f_i}$, that is $f(x'), f(x) \in \mathrm{st}\big(f_i(x), \tau_{f_i}\big)$. Consequently, $f(x'), f(x) \in T_{f_i}$ and $f(x') \in \mathrm{st}\big(f(x), \tau\big)$.

LEMMA 3. *If functions of a family $F \subset Y^X$ are equally continuous, then the natural topology and the topology of point convergence in F are identical.*

Proof. Since the natural topology is finer than the topology of point convergence, it suffices to show that, for each $Z \in K$, $\tau \in C$ and $g \in F$, there exist $x_1, x_2, \ldots, x_k \in X$ and open sets $U_1, U_2, \ldots, U_k \subset Y$ such that

$$g \in F \cap \bigcap\{P(\{x_i\}, U_i)\colon 1 \leqslant i \leqslant k\} \subset \mathrm{st}(g, \hat{\tau}|Z).$$

By Lemma 1, there is $\tau' \in C$ such that

$$\mathrm{st}^{(3)}\big(g(x), \tau'\big) \subset \mathrm{st}\big(g(x), \tau\big) \quad \text{for every } x \in Z.$$

Since functions of F are equally continuous, for every $x \in Z$ there exists an open neighborhood $G_x \ni x$ such that

$$f[G_x] \subset \mathrm{st}\big(f(x), \tau'\big) \quad \text{for each } f \in F.$$

Since Z is compact, $Z \subset \bigcup\{G_{x_i}: 1 \leqslant i \leqslant k\}$.

Write $U_i = \operatorname{Int} \operatorname{st}(g(x_i), \tau')$. If

$$f \in F \cap \bigcap \{P(\{x_i\}, U_i): 1 \leqslant i \leqslant k\} \quad \text{and} \quad x \in Z,$$

then $x \in G_{x_i}$ for some $i \leqslant k$ and

$$f(x) \in \operatorname{st}(f(x_i), \tau'), \quad f(x_i) \in \operatorname{st}(g(x_i), \tau'), \quad g(x) \in \operatorname{st}(g(x_i), \tau').$$

Thus $f(x) \in \operatorname{st}^{(3)}(g(x), \tau') \subset \operatorname{st}(g(x), \tau)$ and, consequently, $f \in \operatorname{st}(g, \hat{\tau}|Z)$, which completes the proof.

Let $\Pi = \mathop{\mathsf{P}}_{x \in X} Y_x$, where $Y_x = Y$ for every $x \in X$, and let p_x denote the projection $p_x: \Pi \to Y_x$.

LEMMA 4. *If functions of a family $F \subset Y^X$ are equally continuous and $p_x[F]$ is compact in Y for every $x \in X$, then $\mathrm{cl}_\Pi F \subset Y^X$, functions of the collection $\mathrm{cl}_\Pi F$ are equally continuous, and $\mathrm{cl}_\Pi F$ is compact both in Π and in the natural topology on Y^X.*

Proof. Let $x \in X$ and $\tau \in C$. Since each semi-uniform covering has a closed semi-uniform refinement and an open semi-uniform refinement [4], there exists an open covering $\tau' \in C$ such that $\{\mathrm{cl}\, T': T' \in \tau'\}$ refines τ. Since $p_x[F]$ is compact,

$$p_x[F] \subset \{T_i' \in \tau': 1 \leqslant i \leqslant k\}.$$

By Lemma 1, there is a $\tau'' \in C$ such that

$$\operatorname{st}(y, \tau'') \subset \bigcup\{T_i': y \in T_i'\} \quad \text{for each } y \in p_x[F].$$

Functions of the family F are equally continuous: thus the point $x \in X$ has an open neighborhood $G_x(\tau)$ such that

$$f[G_x(\tau)] \subset \operatorname{st}(f(x), \tau'') \quad \text{for every } f \in F.$$

Let $V_x(\tau) = \bigcup\{\mathrm{cl}\, T_i' \times \mathrm{cl}\, T_i': 1 \leqslant i \leqslant k\}$. The set

$$\begin{aligned} F_x(\tau) &= \{f \in \Pi: (f(x), f(x')) \in V_x(\tau) \text{ for every } x' \in G_x(\tau)\} \\ &= \bigcap\{\{f \in \Pi: (f(x), f(x')) \in V_x(\tau)\}: x' \in G_x(\tau)\} \end{aligned}$$

is closed in Π. Thus the set $F^* = \bigcap\{F_x(\tau): x \in X, \tau \in C\}$ is also closed in Π. Since $F \subset F^* \subset Y^X$ and functions of F^* are equally continuous, $\mathrm{cl}_\Pi F \subset Y^X$ and functions of $\mathrm{cl}_\Pi F$ are equally continuous.

The set $p_x[F]$ is closed in Y; thus $p_x[\mathrm{cl}_\Pi F] = p_x[F]$. Therefore,

$$\mathrm{cl}_\Pi F \subset \mathop{\mathsf{P}}_{x \in X} p_x[F]$$

and, consequently, $\mathrm{cl}_\Pi F$ is compact in Π. It follows from Lemma 3 that $\mathrm{cl}_\Pi F$ is compact in the natural topology on Y^X.

THEOREM 2. *Let X be a locally compact space. A closed subspace F of the space Y^X with the natural topology is compact iff the following holds:*

(a) *functions of the family F are equally continuous,*

(b) *the set $p_x[F]$ is compact in Y for every $x \in X$.*

Proof. The necessity follows from Lemma 2. It remains to prove the sufficiency. It follows from (a), (b) and Lemma 4 that the family $\mathrm{cl}_\Pi F \subset Y^X$ is compact both in Π and in the space Y^X with the natural topology. By Lemma 3, since F is closed in the natural topology on Y^X, $F = \mathrm{cl}_\Pi F$.

Let $F|Z = \{f|Z\colon f \in F\}$, where $f|Z$ is the restriction of f to Z.

Similarly to Theorem 8.2.5 of [1] it is possible to prove the following theorem:

THEOREM 3. *Let X be a k-space. A closed subspace F of the space Y^X with the natural topology is compact iff the following holds:*

(a) *functions of the family $F|Z$ are equally continuous for every $Z \in K$,*

(b) *the set $p_x[F]$ is compact in Y for every $x \in X$.*

Poppe [2] has introduced a generalization of a uniform space similar to that of a semi-uniform space. He proves Theorem 2 provided that X is a compact space. Rinow [3] also examines a generalization of a uniform space.

REFERENCES

[1] R. Engelking, *Outline of general topology*, Amsterdam - Warszawa 1968.

[2] H. Poppe, *Ein Kompaktheitskriterium für Abbildungsräume mit einer verallgemeinerten uniformen Struktur*, Proceedings of the Second Prague Topological Symposium 1966, Prague 1967, p. 284 - 289.

[3] W. Rinow, *Über die verallgemeinerten uniformen Structuren von Morita und ihre Vervollständigung*, ibidem, p. 297 - 305.

[4] A. K. Steiner and E. F. Steiner, *On semi-uniformities*, Fundamenta Mathematicae 83 (1973), p. 47 - 58.

Reçu par la Rédaction le 17. 5. 1974;
en version modifiée le 24. 9. 1974