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CLOSED MAPPINGS WHICH LOWER DIMENSION

BY

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The purpose of this paper is to prove an improved version of the Hurewicz theorem for closed mappings which lower dimension ([12], Theorem III.8, p. 63-68). The theorem we prove was first proved by Jung [4] for mappings on compact spaces. This paper also provides an alternate proof of the classical Hurewicz theorem in general metric spaces.

Related papers on closed mappings which lower dimension are Jung [3], [4], Lelek [8], [9], Morita [10], Nagami [11], and Skljarenko [13]. We also improve a result of Williams [14]. Other related results are in [5], [6].

Throughout the paper we assume that X and Y are general metric spaces with $f: X \to Y$ a mapping from X onto Y. By $\dim X$ is meant the Lebesgue covering dimension of X. A reference for dimension theory in general metric spaces is Nagata [12].

We first prove a few results which will be needed in the proof of Theorem 4 which is our main result.

1. THEOREM. *If f is a closed mapping and $A_n = \{y \in Y : \dim f^{-1}(y) \geqslant n\}$, then A_n is an F_σ in Y.*

The proof is simplified if a special case is proved first.

2. LEMMA. *If A is a compact subset of the space X and for each $\varepsilon > 0$ there is a finite open covering $\{U_i\}$ of A such that $\operatorname{ord}\{U_i\} \leqslant n+1$ and $\operatorname{diam} U_i < \varepsilon$ for each i, then $\dim A \leqslant n$.*

It is clear that this lemma characterizes the dimension of the compact subsets of a metric space. For a proof see Hurewicz and Wallman [2], Theorem V.8, p. 67.

3. LEMMA. *Let f be closed with compact point inverses and A_n as in Theorem 1. Then A_n is an F_σ in Y.*

Proof. It will be equivalent to show that if $B_n = \{y \in Y : \dim f^{-1}(y) \leqslant n\}$, then B_n is a G_δ in Y. Let $B_{n,k} = \{y \in Y :$ there is an open cover $\{U_i : i = 1, \ldots, p\}$ of $f^{-1}(y)$ with $\operatorname{diam} U_i < 1/k$ and $\operatorname{ord}\{U_i\} \leqslant n+1\}$.

Now $B_{n,k}$ is an open set, for if $y \in B_{n,k}$, then let $\{U_i\}$ be an open cover of $f^{-1}(y)$ satisfying the conditions for y to be in $B_{n,k}$. Then let

$$U = Y - f\Big(X - \bigcup_{i=1}^{p} U_i\Big).$$

Then $y \in U$ and since f is closed, U is open. Now let $z \in U$. Then

$$f^{-1}(z) \subset \bigcup_{i=1}^{p} U_i.$$

Thus $z \in B_{n,k}$ and $U \subset B_{n,k}$. Therefore $B_{n,k}$ is an open set. However, $B_n = \bigcap_{k=1}^{\infty} B_{n,k}$ by Lemma 2 since $f^{-1}(y)$ is compact for each $y \in Y$. Therefore B_n is a G_δ.

Proof of Theorem 1. Let

$$X_0 = \bigcup_{y \in Y} \mathrm{Fr}\big(f^{-1}(y)\big).$$

Then X_0 is closed in X and $f|X_0 : X_0 \to f(X_0)$ is closed with compact point inverses. Thus by Lemma 3 the set $A_n' = \{y \in f(X_0) : \dim f^{-1}(y) \cap X_0 \geqslant n\}$ is an F_σ. Let $A_n'' = \{y \in Y : \dim \mathrm{int} f^{-1}(y) \geqslant n\}$. Then let

$$B = \bigcup_{y \in A_n''} \mathrm{int} f^{-1}(y).$$

Then B is an open set in X which maps onto A_n''. But then B is an F_σ, and thus A_n'' is also an F_σ by the closedness of the mapping f. Now if $y \in A_n$, then either $\dim \mathrm{Fr}\big(f^{-1}(y)\big) \geqslant n$ or $\dim \mathrm{int} f^{-1}(y) \geqslant n$. Therefore $A_n = A_n' \cup A_n''$ and A_n is an F_σ as asserted.

4. THEOREM. *Let f be closed with $\dim X \geqslant n > m \geqslant \dim Y$. Then $n \leqslant \dim A + \dim f$, where $A = \{y \in Y : \dim f^{-1}(y) \geqslant n - m\}$.*

We still need some more results before we can proceed to the proof of Theorem 4. Some of the lemmas that we prove should be of interest in themselves.

5. LEMMA. *Let $\{C_i\}_{i=1}^{\infty}$ be a countable collection of non-empty F_σ's in X with $\dim X = n$ and $\dim C_i = n_i$. Let F and G be disjoint closed sets in X. Then there is an open set U with $F \subset U \subset \overline{U} \subset X - G$ with $\dim \mathrm{Fr}(U) \leqslant n-1$ and $\dim \mathrm{Fr}(U) \cap C_i \leqslant n_i - 1$.*

Proof. Let A_0 be an F_σ in X with $\dim A_0 = 0$ and $\dim(X - A_0) = n-1$. Then let $A_i \subset C_i$ be an F_σ in C_i, hence an F_σ in X, with $\dim A_i = 0$ and with $\dim(C_i - A_i) = n_i - 1$. Then let $A = \bigcup_{i=0}^{\infty} A_i$. Then A is an F_σ and $\dim A = 0$. Let F and G be as stated. Then it is possible to find an open set U with $F \subset U \subset \overline{U} \subset X - G$ such that $\mathrm{Fr}(U) \cap A = \emptyset$. But then $\dim \mathrm{Fr}(U) \leqslant n-1$ and $\dim \mathrm{Fr}(U) \cap C_i \leqslant n_i - 1$.

6. LEMMA. *Let* $\{C_i\}_{i=1}^{\infty}$ *be a countable collection of non-empty* F_σ*'s in* X *with* $\dim X = n$ *and* $\dim C_i = n_i$. *Then if* $\{U_\gamma : \gamma \in \Gamma\}$ *is an open cover of* X, *then there is a locally finite open refinement,* $\{V_a : a \in A\}$ *of* $\{U_\gamma\}$ *such that* $\dim \mathrm{Fr}(V_a) \cap C_i \leqslant n_i - 1$ *for all* i *and all* a *and* $\dim \mathrm{Fr}(V_a) \leqslant n-1$.

Proof. Let $\{W_a : a \in A\}$ be a locally finite open refinement of $\{U_\gamma : \gamma \in \Gamma\}$ and $\{F_a\}$ a closed refinement of $\{W_a\}$ with $F_a \subset W_a$. By lemma 5 there is for each a a V_a open with $F_a \subset V_a \subset \overline{V}_a \subset W_a$ with $\dim \mathrm{Fr}(V_a) \leqslant n-1$ and with $\dim \mathrm{Fr}(V_a) \cap C_i \leqslant n_i - 1$. Then $\{V_a\}$ is the required collection. It refines $\{W_a\}$ and thus $\{U_\gamma\}$ and is a cover since $F_a \subset V_a$ for each a.

7. LEMMA. *Let* $X = C_1 \cup C_2$ *with* C_i *closed in* X. *Let* $f_i : C_i \to S_n$ *for* $i = 1$ *and* 2. *Suppose that* $\dim\{p \in C_1 \cap C_2 : f_1(p) \neq f_2(p)\} \leqslant n-1$. *Then* f_1 *can be extended over all of* X *into* S_n.

Proof. This is in Nagata [12], III.3.C, p. 58.

The reader should be reminded of the characterization of dimension in terms of mappings into spheres ([12], Theorem III.2 and Corollary, p. 53-54). It is this characterization of dimension that will be used in proving Theorem 4.

8. LEMMA. *Let* C *be a closed subset of* X *and suppose that* $g : C \to S_n$. *Let* $\{V_a : a \in A\}$ *be a locally finite open collection in* X *such that for each* a, g *has an extension* g_a *mapping* $C \cup \overline{V}_a$ *into* S_n. *Then if* $\dim \mathrm{Fr}(V_a) \leqslant n-1$ *for all* a, *then* g *has an extension to* $C \cup (\bigcup_{a \in A} \overline{V}_a)$ *into* S_n.

Proof. Let A be well ordered. Let $g_{a_1} : C \cup \overline{V}_{a_1} \to S_n$ be the given extension of g to $C \cup \overline{V}_{a_1}$ for a_1. Suppose that an extension G_β has been constructed for $C \cup (\bigcup_{a \leqslant \beta} \overline{V}_a)$ for all $\beta < \gamma$ such that for $\beta_1 < \beta_2 < \gamma$, $G_{\beta_1} = G_{\beta_2} | C \cup (\bigcup_{a \leqslant \beta_1} \overline{V}_a)$. Then letting $G'_\gamma(x) = G_\beta(x)$ for $x \in C \cup \overline{V}_\beta$, G'_γ will be well defined and continuous on $C \cup (\bigcup_{a<\gamma} \overline{V}_a)$. Note also that $C \cup (\bigcup_{a<\gamma} \overline{V}_a)$ is a closed set. Now let g_γ be the given extension of g to $C \cup \overline{V}_\gamma$. Let $g'_\gamma = g_\gamma | C \cup (\overline{V}_\gamma - \bigcup_{a<\gamma} V_a)$. Then for $B = \{p : g'_\gamma(p) \neq G'_\gamma(p)\}$ we have $B \subset \bigcup_{a \leqslant \gamma} \mathrm{Fr}(V_a)$ and thus $\dim B \leqslant n-1$. Thus by Lemma 7 there is an extension of G'_γ to $C \cup (\bigcup_{a \leqslant \gamma} \overline{V}_a)$. Call this extension G_γ. Then G_γ extends all of the G_β with $\beta < \gamma$. By transfinite induction it is now clear that there is an extension $G : C \cup (\bigcup_{a \in A} \overline{V}_a) \to S_n$ of g.

Proof of Theorem 4. The proof of the theorem is by induction on m. The theorem is vacuously true for $m = -1$. Suppose that it is true for lesser values of m and let $m \geqslant 0$. There are two cases to consider.

Case (i). $\dim f \geqslant n$.

Then $A \neq \emptyset$ and thus $n \leqslant \dim A + \dim f$ since $\dim A \geqslant 0$.

Case (ii). $\dim f \leqslant n-1$.

In this case $n > 0$ and let $g\colon C \to S_{n-1}$ be continuous with C closed in X such that g has no continuous extension to all of X. Such a g exists because $\dim X > n-1$. Since $\dim[f^{-1}(y) \cap C] \leqslant n-1$, by Lemma 7 there is an extension of g to $C \cup f^{-1}(y)$. Thus there is an extension of g to a set $\overline{U}_y$ such that U_y is an open set containing $C \cup f^{-1}(y)$ (see [1], Corollary 53, p. 151). Let $W_y = Y - f(X - U_y)$. Then W_y is open in Y and $y \in W_y$. Let $\{V_a : a \in \Gamma\}$ be a locally finite open refinement of $\{W_y\colon y \in Y\}$ which has the property that (1) $\dim \mathrm{Fr}(V_a) \leqslant m-1$ and (2) $\dim \mathrm{Fr}(V_a) \cap A \leqslant \dim A - 1$ provided $A \neq \emptyset$. This is possible by Lemma 6 and Theorem 1. Let $U_a = f^{-1}(V_a)$. Then $\{U_a\}$ is a locally finite open cover of X and g can be extended to $C \cup \overline{U}_a$. By Lemma 8 there must be some $a \in \Gamma$ such that $\dim \mathrm{Fr}(U_a) \geqslant n-1$ or g would have an extension to all of X. Let $B = f^{-1}(\mathrm{Fr}(V_a))$. Then $\mathrm{Fr}(U_a) \subset B$ and thus $\dim B \geqslant n-1$. Let $f' = f|B : B \to \mathrm{Fr}(V_a)$. Then f' is closed. Let $A^* = \{y \in f'(B) : \dim f'^{-1}(y) \geqslant (n-1)-(m-1)\}$. Clearly $A^* \subset A$. By the induction assumption $A^* \neq \emptyset$ and hence $A \neq \emptyset$. However, $A^* \subset \mathrm{Fr}(V_a)$ also and therefore $A^* \subset A \cap \mathrm{Fr}(V_a)$. This implies that $\dim A^* \leqslant \dim A - 1$. By the induction assumption one has that $n-1 \leqslant \dim A^* + \dim f'$. This leads to $n-1 \leqslant \dim A - 1 + \dim f$ and thus $n \leqslant \dim A + \dim f$.

9. COROLLARY. *If* $\dim X > \dim Y = m$ *and* f *is a closed mapping with* $A = \{y \in Y : \dim f^{-1}(y) \geqslant \dim X - m\}$, *then* $\dim X \leqslant \dim A + \dim f$.

Proof. This is clearly true if $\dim X$ is finite by letting $n = \dim X$ in Theorem 4. If $\dim X$ is infinite, then $\dim f$ must also be infinite by Theorem 4 since $\dim A$ is bounded by m for all n. Thus in this case also $\dim X \leqslant \dim A + \dim f$.

Since we did not use the Hurewicz theorem in the proof we can now state it as a corollary.

10. COROLLARY. *If* f *is closed, then* $\dim X \leqslant \dim Y + \dim f$.

It is also clear from the proof of Theorem 4 that the following theorem holds. This result was first due to Williams [14] for compact X.

11. THEOREM. *If* f *is closed and* $\dim X \geqslant n$ *with* $\dim Y \leqslant m$, *then there is a closed set* $K \subset Y$ *with* $\dim K \leqslant m-1$ *and with* $\dim f^{-1}(K) \geqslant n-1$.

12. COROLLARY. *If* f *is closed and* $\dim X \geqslant n > m \geqslant \dim Y$, *then for every* $0 \leqslant k \leqslant m$, *there is a closed set* $K \subset Y$ *with* $\dim K \leqslant k$ *and with* $\dim f^{-1}(K) \geqslant k + n - m$.

This last result is a generalization of the Hurewicz theorem in a certain sense.

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