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ON ANALYTIC INVARIANT MEASURES FOR EXPANDING MAPPINGS

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It was stated without proof in [1] that an analytic expanding mapping of an analytic compact connected manifold has a unique analytic normalized invariant measure. In this note we give the complete proof of that theorem.

We first recall some definitions. Let M be a connected C^∞-manifold with a fixed Riemannian C^∞-metric $\|\cdot\|$. Then a Borel measure μ on M is said to be C^∞ if for each local chart (G, a) on M there exists a positive C^∞-function g_a on $a(G)$ such that

$$\mu(A) = \int_{a(A)} g_a(x)\,dx$$

for each Borel set $A \subset G$. If μ is a C^∞-measure on M and $\varphi : M \to M$ is a C^∞-immersion, then there exists a unique positive C^∞-function $J\varphi$ on M such that

$$\mu(\varphi(A)) = \int_A J\varphi\,d\mu$$

for each Borel set $A \subset M$, provided that $\varphi|A$ is injective. A C^1-mapping φ: $M \to M$ is called *expanding* if there exist numbers $a > 0$ and $b > 1$ such that

$$\|D\varphi^n(\mathfrak{a})\| \geqslant ab^n\|\mathfrak{a}\|$$

for $\mathfrak{a} \in T(M)$ and $n \in N$. From now on, M will be compact.

The following theorem will be proved:

THEOREM. *For each expanding C^∞-mapping φ of M there exists a unique φ-invariant normalized C^∞-measure.*

Proof. In view of [2] it is sufficient to prove only the existence. For this purpose let μ be the normalized Riemannian measure induced by $\|\cdot\|$. Moreover, let U: $C^\infty(M, R) \to C^\infty(M, R)$ be the operator such that

$$U(f)(x) = \sum_{\bar{x}\in\varphi^{-1}(x)} f(\bar{x})\big(J\varphi(\bar{x})\big)^{-1} \tag{1}$$

for $x \in M$. It is well known that if g is a positive C^∞-function on M, then the C^∞-measure with the density g with respect to μ is φ-invariant iff

$$U(g) = g. \tag{2}$$

We shall prove the existence of such a function using the universal covering space of M. It is well known that two C^∞-diffeomorphic C^ω-manifolds are C^ω-diffeomorphic. Hence, in view of [3], there exists a C^ω-covering mapping $\pi\colon \boldsymbol{R}^m \to M$, where $m = \dim M$. Let $\|\cdot\|_*$ be the lifting of $\|\cdot\|$ to $\boldsymbol{R}^m$, and let μ_* be the Riemannian measure induced by $\|\cdot\|_*$. It is well known that there exists a C^ω-diffeomorphism φ_* of $\boldsymbol{R}^m$ onto itself such that

$$\pi\circ\varphi_* = \varphi\circ\pi. \tag{3}$$

Moreover, φ_* is expanding. Substituting an iteration of φ for φ, one may assume that there exists $b > 1$ such that

$$\|D\varphi_*(\mathfrak{a})\|_* \geqslant b\,\|\mathfrak{a}\|_* \tag{4}$$

for $\mathfrak{a} \in T(\boldsymbol{R}^m)$. Let Γ be the group of deck transformations of the covering mapping π. Each element of Γ is a C^ω-isometry. Let $C^\omega_*(\boldsymbol{R}^m, \boldsymbol{R})$ be the set of all Γ-invariant $f \in C^\omega(\boldsymbol{R}^m, \boldsymbol{R})$. Then $f \to f_* = f\circ\pi$ is a bijection of $C^\omega(M, \boldsymbol{R})$ onto $C^\omega_*(\boldsymbol{R}^m, \boldsymbol{R})$. Therefore

$$U_*(f_*) = \big(U(f)\big)_*, \tag{5}$$

where $f \in C^\omega(M, \boldsymbol{R})$, defines the operator of $C^\omega_*(\boldsymbol{R}^m, \boldsymbol{R})$ into itself. We shall express U_* by means of φ_* and some elements of Γ. For this purpose let us remark that, by (3), $\gamma \xrightarrow{\overline{\varphi}} \varphi_*\circ\gamma\circ\varphi_*^{-1}$ is a homomorphism of Γ onto its subgroup Γ_1. Let Γ_2 be a set such that its intersection with each right coset of Γ_1 in Γ is a one-point set. Then

$$U_*(f) = \sum_{\gamma\in\Gamma_2} f\circ\varphi_*^{-1}\circ\gamma(J\varphi_*^{-1})\circ\gamma \tag{6}$$

for $f \in C^\omega_*(\boldsymbol{R}^m, \boldsymbol{R})$. For the proof, we first show that

$$\Gamma_2 \ni \gamma \to (\pi\circ\varphi_*^{-1}\circ\gamma)(x) \in \varphi^{-1}\big(\pi(x)\big) \tag{7}$$

is a bijection for $x \in \boldsymbol{R}^m$. For this purpose let $x \in \boldsymbol{R}^m$. Then (3) implies $(\pi\circ\varphi_*^{-1}\circ\gamma)(x) \in \varphi^{-1}\big(\pi(x)\big)$. Moreover, the mapping defined in (7) is injective. To prove this, let

$$(\pi\circ\varphi_*^{-1}\circ\gamma_1)(x) = (\pi\circ\varphi_*^{-1}\circ\gamma_2)(x)$$

for some $\gamma_1, \gamma_2 \in \Gamma_2$. Since Γ acts transitively on each fibre of the covering mapping π and without fixed points, there exists $\gamma \in \Gamma$ such that

$$\varphi_*\circ\gamma\circ\varphi_*^{-1}\circ\gamma_1 = \gamma_2.$$

By the definition of Γ_2, this implies that $\gamma_1 = \gamma_2$. To complete the proof of (7) it is sufficient to observe that

(8) *the index of Γ_1 in Γ is equal to the multiplicity of φ.*

For the proof, let $\varphi_{\bar{x}}$ be the canonical homomorphism of $\pi_1(M, \bar{x})$ into $\pi_1(M, \varphi(\bar{x}))$ for $\bar{x} \in M$ and let $f_{\bar{\bar{x}}}$ be the canonical isomorphism of Γ onto $\pi_1(M, \pi(\bar{\bar{x}}))$ for $\bar{\bar{x}} \in \boldsymbol{R}^m$. Then $f_{\varphi_*(x)} \circ \bar{\varphi} = \varphi_{\pi(x)} \circ f_x$ and the multiplicity of φ is equal to the index of $\varphi_{\pi(x)}\big(\pi_1(M, \pi(x))\big)$ in $\pi_1\big(M, \varphi(\pi(x))\big)$ for $x \in \boldsymbol{R}^m$. This implies (8).

Returning to the proof of (6), let us remark that since π is a local isometry, (3) implies $J\varphi_* = (J\varphi)\circ\pi$. Hence

$$J\varphi_*^{-1} = \big((J\varphi)\circ\pi\circ\varphi_*^{-1}\big)^{-1}. \tag{9}$$

Now (1), (5), (7), and (9) give (6).

Using (6) one proves by induction that

$$U_*^n(f) = \tag{10}$$
$$= \sum_{(\gamma_1,\ldots,\gamma_n)\in\Gamma_2^n} f\circ\varphi_*^{-1}\circ\gamma_n\circ\ldots\circ\varphi_*^{-1}\circ\gamma_1 \prod_{i=1}^{n} (J\varphi_*^{-1})\circ\gamma_i\circ\varphi_*^{-1}\circ\gamma_{i-1}\circ\ldots\circ\varphi_*^{-1}\circ\gamma_1$$

for $f \in C_*^\omega(\boldsymbol{R}^m, \boldsymbol{R})$ and $n \in \boldsymbol{N}$.

Now let $x_0 \in \boldsymbol{R}^m$ be a fixed point of φ_* (see [3]). Then there exists $r_1 > 0$ such that

$$\pi(G_1) = M, \tag{11}$$

where $G_1 = B_{\varrho_*}(x_0, r_1)$ and ϱ_* is the distance induced by $\|\cdot\|_*$. We prove now that there exists $r_2 \geqslant r_1$ such that

$$(\gamma_n\circ\varphi_*^{-1}\circ\gamma_{n-1}\circ\ldots\circ\varphi_*^{-1}\circ\gamma_1)(G_1) \subset G_2, \tag{12}$$

$$(\varphi_*^{-1}\circ\gamma_n\circ\ldots\circ\varphi_*^{-1}\circ\gamma_1)(G_1) \subset G_2 \tag{13}$$

for $\gamma_1, \ldots, \gamma_n \in \Gamma_2$ and $n \in \boldsymbol{N}$, where $G_2 = B_{\varrho_*}(x_0, r_2)$. For this purpose let us remark that (4) implies

$$\varrho_*\big(\varphi_*^{-1}(x), \varphi_*^{-1}(y)\big) \leqslant b^{-1}\varrho_*(x, y) \quad \text{for } x, y \in \boldsymbol{R}^m. \tag{14}$$

By (14), it suffices to prove (12) which in turn follows from

$$\varrho_*\big((\gamma_n\circ\varphi_*^{-1}\circ\gamma_{n-1}\circ\ldots\circ\varphi_*^{-1}\circ\gamma_1)(x), x_0\big) \leqslant r_0 \sum_{i=0}^{n-1} b^{-i} \tag{15}$$

for $x \in G_1$, where $\gamma_1, \ldots, \gamma_n \in \Gamma_2$, $n \in \boldsymbol{N}$, and

$$r_0 = \sup_{\gamma\in\Gamma_2} \sup_{x\in G_1} \varrho_*\big(\gamma(x), x_0\big).$$

One proves (15) by induction using the fact that x_0 is a fixed point of φ_* and (14). Now, for $((\mathfrak{a}, \mathfrak{b}), (x, y)) \in T((\boldsymbol{R}^m)^2)$ let

$$\|((\mathfrak{a}, \mathfrak{b}), (x, y))\|_0 = \sqrt{\|(\mathfrak{a}, x)\|_*^2 + \|(\mathfrak{b}, x)\|_*^2}. \tag{16}$$

Then $\|\cdot\|_0$ is a Riemannian C^∞-metric on $(\boldsymbol{R}^m)^2$. Since the Riemannian metric $\|\cdot\|_*$ is complete, $\bar{G}_2$ is compact. Therefore, $\gamma \in \Gamma_2$, φ_*^{-1}, and $J\varphi_*^{-1}$ restricted to $\bar{G}_2$ have the complex analytic extensions f_γ, g, and h to $B_{\varrho_0}(G_2, \delta_2) \subset (\boldsymbol{R}^m)^2 = \boldsymbol{C}^m$, respectively, where δ_2 is a positive number and ϱ_0 is the distance induced by $\|\cdot\|_0$. Now let us remark that

$$Df_\gamma(x) = D\gamma(x) \oplus D\gamma(x), \qquad Dg(x) = D\varphi_*^{-1}(x) \oplus D\varphi_*^{-1}(x) \tag{17}$$

for $x \in \bar{G}_2$, where $\gamma \in \Gamma_2$. From (4), (16), and (17) it follows that

$$\|Df_\gamma(x)\|_0 = 1 \quad \text{and} \quad \|Dg(x)\|_0 \leqslant b^{-1}$$

for $x \in \bar{G}_2$, where $\gamma \in \Gamma_2$. Hence, by possible decreasing δ_2, one may assume that there exist positive numbers d_1, d_2, $d_1 d_2 < 1$, such that

$$\|Df_\gamma(z)\|_0 \leqslant d_1 \quad \text{and} \quad \|Dg(z)\|_0 \leqslant d_2 \tag{18}$$

for $z \in B_{\varrho_0}(G_2, \delta_2)$, where $\gamma \in \Gamma_2$. Decreasing δ_2 again, if necessary, one shows that there is $L > 0$ such that

$$J\varphi_*^{-1}(x) \geqslant L^{-1} \tag{19}$$

for $x \in G_2$ and

$$|J\varphi_*^{-1}(x) - h(z)| \leqslant L\varrho_0(x, z) \tag{20}$$

for $x \in G_2$ and $z \in B_{\varrho_0}(G_2, \delta_2)$. We now prove that there exists $\delta_1 \in]0, \delta_2]$ such that if $z \in B_{\varrho_0}(G_1, \delta_1)$, $\gamma_1, \ldots, \gamma_n \in \Gamma_2$, and $n \in \boldsymbol{N}$, then

(21) $f_{\gamma_n}\Big(g\big(f_{\gamma_{n-1}} \cdots g(f_{\gamma_1}(z)) \ldots\big)\Big)$ *is well defined and belongs to* $B_{\varrho_0}(G_2, \delta_2)$.

For this purpose let $\gamma_1, \ldots, \gamma_n \in \Gamma_2$ and $n \in \boldsymbol{N}$. Then, in view of (12) and (13), we have

(22) *if* $x \in G_1$, *then*

$$f_{\gamma_n}\Big(g\big(f_{\gamma_{n-1}} \cdots g(f_{\gamma_1}(x)) \ldots\big)\Big) = \gamma_n\Big(\varphi_*^{-1}\big(\gamma_{n-1} \cdots \varphi_*^{-1}(\gamma_1(x)) \ldots\big)\Big).$$

Now let $z \in B_{\varrho_0}(G_1, \delta_1)$, where $\delta_1 = \delta_2 d_1^{-1}$. Then there exists $x \in G_1$ such that $\varrho_0(z, x) < \delta_1$. Let p be a C^∞-path from x to z such that its $\|\cdot\|_0$-length is less than δ_1. Then, by (12) and (22), it is sufficient to prove that

(23) *the path* $t \to f_{\gamma_n}\Big(g\big(f_{\gamma_{n-1}} \cdots g(f_{\gamma_1}(p(t))) \ldots\big)\Big)$ *is well defined and its* $\|\cdot\|_0$-*length is less than* $\delta_2 (d_1 d_2)^{n-1}$.

One shows (23) by induction on n using (12), (13), (18), and (22).

We prove now that there exists $c > 0$ such that for each $z \in B_{\varrho_0}(G_1, \delta_1)$ there exists $x \in G_1$ such that

$$(24)\qquad \prod_{i=1}^{n} \frac{\left|h\left(f_{\gamma_i}\left(g\left(f_{\gamma_{i-1}} \cdots g\left(f_{\gamma_1}(z)\right)\ldots\right)\right)\right)\right|}{J\varphi_*^{-1}\left(\gamma_i\left(\varphi_*^{-1}\left(\gamma_{i-1} \cdots \varphi_*^{-1}\left(\gamma_1(x)\right)\ldots\right)\right)\right)} \leqslant c$$

for $\gamma_1, \ldots, \gamma_n \in \Gamma_2$ and $n \in N$. For this purpose, let z and x be such as in the proof of (23). Then, in view of (12) and (19)-(23), the left-hand side of (24) does not exceed

$$\prod_{i=1}^{n}\left(1+\frac{\left|h\left(f_{\gamma_i}\left(g\left(f_{\gamma_{i-1}}\cdots g\left(f_{\gamma_1}(z)\right)\ldots\right)\right)\right)-J\varphi_*^{-1}\left(\gamma_i\left(\varphi_*^{-1}\left(\gamma_{i-1}\cdots\varphi_*^{-1}\left(\gamma_1(x)\right)\ldots\right)\right)\right)\right|}{J\varphi_*^{-1}\left(\gamma_i\left(\varphi_*^{-1}\left(\gamma_{i-1}\cdots\varphi_*^{-1}\left(\gamma_1(x)\right)\ldots\right)\right)\right)}\right)$$

$$\leqslant \exp\left[L^2\sum_{i=1}^{n}\varrho_0\left(f_{\gamma_i}\left(g\left(f_{\gamma_{i-1}}\cdots g\left(f_{\gamma_1}(z)\right)\ldots\right)\right),\ \gamma_i\left(\varphi_*^{-1}\left(\gamma_{i-1}\cdots\varphi_*^{-1}\left(\gamma_1(x)\right)\ \ldots\right)\right)\right)\right]$$

$$\leqslant \exp\left[L^2\sum_{i=1}^{n}\delta_2(d_1d_2)^{n-1}\right].$$

This completes the proof of (24).

Now it is sufficient to show that there exists $d > 0$ such that

$$(25)\qquad d^{-1} \leqslant U_*^n(1) \leqslant d \quad \text{for } n \in N.$$

Suppose that (25) is true. Then, from (10), (12), (24), and (25) it follows that the sequence (F_n) is uniformly bounded, where

$$(26)\qquad F_n(z) = \sum_{(\gamma_1,\ldots,\gamma_n)\in\Gamma_2^n}\ \prod_{i=1}^{n} h\left(f_{\gamma_i}\left(g\left(f_{\gamma_{i-1}}\cdots g\left(f_{\gamma_1}(z)\right)\ldots\right)\right)\right)$$

for $z \in B_{\varrho_0}(G_1, \delta_1)$ and $n \in N$. Therefore, by the Montel theorem for complex analytic functions, there exists an increasing sequence (k_n) of natural numbers such that the sequence

$$\left(k_n^{-1}\sum_{i=0}^{k_n-1} F_i\right)$$

is locally uniformly convergent to a complex analytic function ($F_0 = 1$). From (10), (22), and (26) it follows that F_n is the extension of $U_*^n(1)\,|\bar{G}_1$ for $n \in N$. Therefore, the sequence

$$\left(k_n^{-1}\sum_{i=0}^{k_n-1} U_*^i(1)\,|G_1\right)$$

is uniformly convergent to a C^∞-function. Hence, by (11), the sequence

$$\Big(k_n^{-1}\sum_{i=0}^{k_n-1} U_*^i(1)\Big)$$

is uniformly convergent to a function $f \in C_*^\infty(\boldsymbol{R}^m, \boldsymbol{R})$. From (25) it follows that

$$U_*(f) = f \tag{27}$$

and f is positive. But there exists a positive function $g \in C^\infty(M, \boldsymbol{R})$ such that $f = g_*$. Therefore (27) implies (2), which will complete the proof. It remains to prove (25). For this purpose let us remark that the method of the proof of (24) also enables us to show the existence of $d > 0$ such that

$$\prod_{i=1}^{n} \frac{J\varphi_*^{-1}\Big(\gamma_i\Big(\varphi_*^{-1}\big(\gamma_{i-1} \ldots \varphi_*^{-1}(\gamma_1(x))\ldots\big)\Big)\Big)}{J\varphi_*^{-1}\Big(\gamma_i\Big(\varphi_*^{-1}\big(\gamma_{i-1} \ldots \varphi_*^{-1}(\gamma_1(y))\ldots\big)\Big)\Big)} \leqslant d \tag{28}$$

for $x, y \in G_1$, where $\gamma_1, \ldots, \gamma_n \in \Gamma_2$ and $n \in \boldsymbol{N}$. It is easy to see that there exists a Borel set $A \subset G_1$ such that

(29) $\pi | A$ *is a bijection.*

Let $x \in A$, $\gamma_1, \ldots, \gamma_n \in \Gamma_2$, and $n \in \boldsymbol{N}$. Then (28) implies

$$\begin{aligned} (30)\qquad & d\mu_*\big((\varphi_*^{-1}\circ\gamma_n\circ \ldots \circ\varphi_*^{-1}\circ\gamma_1)(A)\big) \\ &= d\int_A \prod_{i=1}^{n} J\varphi_*^{-1}\Big(\gamma_i\Big(\varphi_*^{-1}\big(\gamma_{i-1} \ldots \varphi_*^{-1}(\gamma_1(y))\ldots\big)\Big)\Big)\,d\mu_*(y) \\ &\geqslant \mu_*(A)\prod_{i=1}^{n} J\varphi_*^{-1}\Big(\gamma_i\Big(\varphi_*^{-1}\big(\gamma_{i-1} \ldots \varphi_*^{-1}(\gamma_1(x))\ldots\big)\Big)\Big). \end{aligned}$$

But from (7) it follows that for $x \in \boldsymbol{R}^m$ and $n \in \boldsymbol{N}$ the mapping

$$\Gamma_2^n \ni (\gamma_1, \ldots, \gamma_n) \to (\pi\circ\varphi_*^{-1}\circ\gamma_n\circ \ldots \circ\varphi_*^{-1}\circ\gamma_1)(x) \in \varphi^{-n}\big(\pi(x)\big)$$

is a bijection. Hence, form (29) we infer that for $n \in \boldsymbol{N}$

(31) $\pi\Big|\bigcup_{(\gamma_1,\ldots,\gamma_n)\in\Gamma_2^n} (\varphi_*^{-1}\circ\gamma_n\circ \ldots \circ\varphi_*^{-1}\circ\gamma_1)(A)$ *is a bijection*

and

(32) *the sets* $(\varphi_*^{-1}\circ\gamma_n\circ \ldots \circ\varphi_*^{-1}\circ\gamma_1)(A)$ *are pairwise disjoint, where* $(\gamma_1, \ldots, \gamma_n) \in \Gamma_2^n$.

It follows from (29) and (31) that

$$\mu_*(A) = \mu_*\Big(\bigcup_{(\gamma_1,\ldots,\gamma_n)\in\Gamma_2^n} (\varphi_*^{-1}\circ\gamma_n\circ \ldots \circ\varphi_*^{-1}\circ\gamma_1)(A)\Big) = 1.$$

Consequently, by (30) and (32), the right inequality in (25) holds. The proof of the left one is similar. This completes the proof of the Theorem.

Remark 1. The above proof can also be carried over without using the theorem on universal covering mapping of M. But then the proof is a little longer.

Remark 2. We want this note to be self-contained, and therefore we do not use any theorem on convergence to invariant measures for expanding mappings. Otherwise, the last part of the proof could be shortened.

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