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ON ENTIRE FUNCTIONS WHICH TRANSFORM STRAIGHT LINES INTO PARABOLAS

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The well-known principle of circle-transformation of a linear rational function (see [1], p. 160) says:

Suppose that $w = f(z) \not\equiv$ const *is meromorphic in* $|z| < +\infty$. *Then* $w = f(z)$ *transforms circles on the z-plane into circles on the w-plane, including straight lines among circles, if and only if f is a linear rational function.*

The purpose of this note is to discuss a problem similar to this principle, i.e., to prove the following

THEOREM. *Suppose that* $w = f(z) \not\equiv$ const *is an entire function of z. Then* $w = f(z)$ *transforms straight lines on the z-plane into parabolas on the w-plane, including doubled half-closed straight lines among parabolas, if and only if f is a quadratic function of z.*

Proof. Let D be the domain where $f'(z) \neq 0$. Since $f \not\equiv$ const, D is not empty. Suppose that c is an arbitrarily fixed point belonging to D.

Let E denote the following set on the real line:

$$E = \{\theta \mid \operatorname{Re}(\exp(i\theta)f'(c)) \neq 0\}. \tag{1}$$

Since c belongs to D, we have $f'(c) \neq 0$. If we put $a = \arg\{f'(c)\}$, then we have

$$\operatorname{Re}[\exp\{i(-a)\}f'(c)] = \operatorname{Re}\{\exp(-ia)|f'(c)|\exp(ia)\} = |f'(c)| \neq 0.$$

Therefore, $-a \in E$. Hence the set E is not empty. E is open, since $\operatorname{Re}\{\exp(i\theta)f'(c)\}$ is continuous on $-\infty < \theta < +\infty$.

We put

$$f\{c + t\exp(i\varphi)\} = p(t) + iq(t), \tag{2}$$

where t is a real variable, φ is an arbitrarily fixed real number belonging to E, and $p(t)$ and $q(t)$ are real-valued functions of t on $-\infty < t < +\infty$.

Differentiating both sides of (2) with respect to t yields

$$\exp(i\varphi)f'\{c+t\exp(i\varphi)\} = p'(t)+iq'(t). \tag{3}$$

Since $\varphi \in E$, by (1) and (3) we have

$$p'(0) = \mathrm{Re}\{\exp(i\varphi)f'(c)\} \neq 0.$$

Hence, by the continuity of $p'(t)$ at $t = 0$, there exists an open interval $-\delta < t < \delta$, where δ is a positive real constant such that $p'(t) \neq 0$ on $-\delta < t < \delta$. Consequently, $p(t)$ is a strictly monotonic function on $-\delta < t < \delta$. Hence, if we put

$$x = p(t) \quad \text{and} \quad y = q(t), \tag{4}$$

then (4) defines a one-valued function $y = h(x)$ of x whose domain, by the definition, is a certain open interval I.

By (2) we see that $p(t)$ and $q(t)$ are infinitely many times differentiable on $-\infty < t < +\infty$. Hence, on I ($\varphi \in E$) we have

$$d^2y/dx^2 = \{p'(t)q''(t)-p''(t)q'(t)\}/p'(t)^3. \tag{5}$$

Differentiating both sides of (3) with respect to t yields

$$\exp(2i\varphi)f''\{c+t\exp(i\varphi)\} = p''(t)+iq''(t). \tag{6}$$

By (3) and (6) we have

$$p'(t)q''(t)-p''(t)q'(t) = \mathrm{Im}[\exp(i\varphi)\overline{f'\{c+t\exp(i\varphi)\}}f''\{c+t\exp(i\varphi)\}]. \tag{7}$$

By (5) and (7), on I ($\varphi \in E$) we have

$$d^2y/dx^2 = \mathrm{Im}\{\exp(i\varphi)\bar{f}'f''\}/p'(t)^3, \tag{8}$$

where we abbreviate $f'\{c+t\exp(i\varphi)\}$ and $f''\{c+t\exp(i\varphi)\}$ to f' and f'', respectively.

Differentiating both sides of (8) with respect to t and taking into account the fact that $\mathrm{Im}(\bar{f}''f'') = 0$, on I ($\varphi \in E$) we have

$$d^3y/dx^3 = [p'(t)\mathrm{Im}\{\exp(2i\varphi)\bar{f}'f'''\}-3p''(t)\mathrm{Im}\{\exp(i\varphi)\bar{f}'f''\}]/p'(t)^5, \tag{9}$$

where f' and f'' are as in (8), and by f''' we denote $f'''\{c+t\exp(i\varphi)\}$.

Differentiating both sides of (9) with respect to t and simplifying the resulting equality yield on I ($\varphi \in E$) the equation

$$\begin{aligned} d^4y/dx^4 = [&p'(t)^2\mathrm{Im}\{\exp(i\varphi)\bar{f}''f'''+\exp(3i\varphi)\bar{f}'f^{(4)}\}- \\ &-7p'(t)p''(t)\mathrm{Im}\{\exp(2i\varphi)\bar{f}'f'''\}- \\ &-3p'(t)p'''(t)\mathrm{Im}\{\exp(i\varphi)\bar{f}'f''\}+ \\ &+15p''(t)^2\mathrm{Im}\{\exp(i\varphi)\bar{f}'f''\}]/p'(t)^7, \end{aligned} \tag{10}$$

where f', f'' and f''' are as in (8) and (9), and by $f^{(4)}$ we denote $f^{(4)}\{c+t\exp(i\varphi)\}$.

By hypothesis, the graph of $y = h(x)$ $(x \in I)$ is a parabolic arc. Hence, by a result proved in [2], we see that $y = h(x)$ satisfies on I

(11) $$3(d^2y/dx^2)(d^4y/dx^4) = 5(d^3y/dx^3)^2.$$

Substituting (8), (9) and (10) into (11), on $-\delta < t < \delta$ $(\varphi \in E)$ we have

(12) $$\begin{aligned} &3\,\mathrm{Im}\{\exp(i\varphi)\bar{f}'f''\}[p'(t)^2\,\mathrm{Im}\{\exp(i\varphi)\bar{f}''f''' + \exp(3i\varphi)\bar{f}'f^{(4)}\} - \\ &-7p'(t)p''(t)\,\mathrm{Im}\{\exp(2i\varphi)\bar{f}'f'''\} - 3p'(t)p'''(t)\,\mathrm{Im}\{\exp(i\varphi)\bar{f}'f''\} + \\ &+15p''(t)^2\,\mathrm{Im}\{\exp(i\varphi)\bar{f}'f''\}] - 5[p'(t)\,\mathrm{Im}\{\exp(2i\varphi)\bar{f}'f'''\} - \\ &-3p''(t)\,\mathrm{Im}\{\exp(i\varphi)\bar{f}'f''\}]^2 = 0, \end{aligned}$$

where f', f'', f''' and $f^{(4)}$ are as in (8), (9) and (10).

Putting $t = 0$ in (12), we obtain $(\varphi \in E)$

(13) $$\begin{aligned} &3\,\mathrm{Im}\{\exp(i\varphi)\overline{f'(c)}f''(c)\}[p'(0)^2\,\mathrm{Im}\{\exp(i\varphi)\overline{f''(c)}f'''(c) + \\ &+\exp(3i\varphi)\overline{f'(c)}f^{(4)}(c)\} - 7p'(0)p''(0)\,\mathrm{Im}\{\exp(2i\varphi)\overline{f'(c)}f'''(c)\} - \\ &-3p'(0)p'''(0)\,\mathrm{Im}\{\exp(i\varphi)\overline{f'(c)}f''(c)\} + \\ &+15p''(0)^2\,\mathrm{Im}\{\exp(i\varphi)\overline{f'(c)}f''(c)\}] - \\ &-5[p'(0)\,\mathrm{Im}\{\exp(2i\varphi)\overline{f'(c)}f'''(c)\} - \\ &-3p''(0)\,\mathrm{Im}\{\exp(i\varphi)\overline{f'(c)}f''(c)\}]^2 = 0. \end{aligned}$$

Putting $t = 0$ in (3) and (6), we have $(\varphi \in E)$

(14) $$p'(0) = \mathrm{Re}\{\exp(i\varphi)f'(c)\},$$

(15) $$p''(0) = \mathrm{Re}\{\exp(2i\varphi)f''(c)\}.$$

Differentiating both sides of (6) with respect to t and putting $t = 0$ in the resulting equality yield

(16) $$p'''(0) = \mathrm{Re}\{\exp(3i\varphi)f'''(c)\}.$$

By (14), (15) and (16) and using the formulas $\mathrm{Re}(A) = (1/2)(A+\bar{A})$, $\mathrm{Im}(A) = (1/2i)(A-\bar{A})$ (A complex), we see that the left-hand side of (13) is a trigonometric polynomial in φ of order 6 if we consider φ as a real variable. Let the coefficient of $\exp(6i\varphi)$ of this trigonometric polynomial be a_6. Then, after some computations, we infer that

(17) $$\begin{aligned} a_6 &= (3\bar{f}'f''/2i)[(f'/2)^2\{\bar{f}'f^{(4)}/2i\} - 7(f'/2)(f''/2)\{\bar{f}'f'''/2i\} - \\ &\quad -3(f'/2)(f'''/2)\{\bar{f}'f''/2i\} + 15(f''/2)^2\{\bar{f}'f''/2i\}] - \\ &\quad -5[(f'/2)^2\{\bar{f}'f'''/2i\}^2 - 2(f'/2)\{\bar{f}'f'''/2i\}\cdot 3(f''/2)\{\bar{f}'f''/2i\} + \\ &\quad +9(f''/2)^2\{\bar{f}'f''/2i\}^2] \\ &= -(1/16)|f'(c)|^4\{3f''(c)f^{(4)}(c) - 5f'''(c)^2\}. \end{aligned}$$

Since (13) holds on the non-empty open set E on the real line, by the Identity Theorem, (13) holds for all complex φ. Consequently, (13) holds for all real φ. Since the representation of a trigonometric polynomial is unique, by (13) we have

$$a_6 = 0. \tag{18}$$

Since c belongs to D, we have

$$f'(c) \neq 0. \tag{19}$$

By (17), (18) and (19), we obtain

$$3f''(c)f^{(4)}(c) - 5f'''(c)^2 = 0. \tag{20}$$

Thus, in D we have

$$3f''(z)f^{(4)}(z) - 5f'''(z)^2 = 0, \tag{21}$$

since c in (20) is an arbitrarily fixed point belonging to D.

By the Identity Theorem we see that (21) holds in $|z| < +\infty$.

Next, we prove that in $|z| < +\infty$

$$f''(z) = \text{const}. \tag{22}$$

The proof is by contradiction. Assume contrary. Then, since $f''(z)$ is a non-constant entire function of z, we can write the power series expansion of $f''(z)$ in $|z| < +\infty$ in the form

$$f''(z) = b_0 + b_p z^p + \dots, \tag{23}$$

where p is a positive integer, and b_0, b_p are complex constants with $b_p \neq 0$.

Substituting (23) into (21) and equating the coefficients of z^{2p-2} of both sides of the resulting equality, we obtain

$$3p(p-1)b_p^2 - 5(pb_p)^2 = 0 \quad \text{or} \quad p(-2p-3)b_p^2 = 0 \quad \text{or} \quad b_p = 0,$$

which contradicts the fact that $b_p \neq 0$.

By (22), in $|z| < +\infty$ we have

$$f(z) = \alpha z^2 + \beta z + \gamma, \tag{24}$$

where α, β and γ are complex constants.

Since $w = f(z)$ transforms straight lines on the z-plane into parabolas on the w-plane, including doubled half-closed straight lines among parabolas, α in (24) must be non-zero. Thus, the "only if" part of the Theorem is proved. The proof of the "if" part of the Theorem is clear.

REFERENCES

[1] Z. Nehari, *Conformal mapping*, New York 1952.
[2] M. R. Perrin, *Sur quelques conséquences géométriques de l'équation différentielle des coniques*, Bulletin de la Société Mathématique de France 31 (1903), p. 54-64.

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