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THE SCHAUDER FIXED-POINT THEOREM FOR CONNECTIVITY MAPS

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In [2]-[4] Cellina showed that, under suitable conditions, a multivalued map could be approximated by a continuous function. This leads to a number of fixed-point theorems for multivalued maps. The construction of the continuous function was based upon a partition of unity, subordinate to an open cover of the domain space.

We use* the technique of Cellina to approximate connectivity functions by continuous functions. Our approximation leads to a generalization, to absolute neighborhood retracts, of Hamilton's theorem that a connectivity function of an n-cell into itself has a fixed point [5], and to the corresponding Stallings generalization to polyhedra [9]. Thus we give a partial answer to question 7 in [9]. For example, it is shown that if $f: X \to X$ is a connectivity function, when X is a compact metric ANR without local cut points and with the fixed-point property for continuous functions, then f has a fixed point. It follows that the Schauder fixed-point theorem carries over to connectivity maps. Specifically, if $f: X \to X$ is a connectivity function, X a closed convex subset of a Banach space and $\overline{f(X)}$ compact, then f has a fixed point.

Let $f: X \to Y$ be a function from a topological space X to a topological space Y. For $C \subseteq X$ the graph over C is defined to be the set $\{(x, f(x)): x \in C\}$, a subspace of the topological space $X \times Y$. The graph of f, denoted by Γf, is defined to be the graph over X.

Definition 1. A function $f: X \to Y$ is called a *connectivity function* if the graph over each connected subset of X is a connected set.

As with the Hamilton and Stallings proofs, it is necessary for us to introduce alternate classes of functions. The boundary of a set U will be denoted by $\mathrm{bd}(U)$.

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Definition 2 (Hamilton [5]). A function $f\colon X \to Y$ is *peripherally continuous* if for each $x \in X$, each open set $V \subseteq X$ for which $x \in V$, and each open set $Q \subseteq Y$ for which $f(x) \in Q$ there exists a neighborhood U of x such that $\bar{U} \subseteq V$ and $f(\mathrm{bd}(U)) \subseteq Q$.

Definition 3 (Stallings [9]). A function $f\colon X \to Y$ is *almost continuous* if for every open subset U of $X \times Y$ with $\Gamma f \subseteq U$ there exists a continuous function $g\colon X \to Y$ such that $\Gamma g \subseteq U$.

Definition 4 (Borsuk [1]). A set X is said to be *unicoherent* mod A if for each pair of connected closed sets M and N of X such that $X = M \cup N$ and $A \cap N = \emptyset$ it is true that $M \cap N$ is connected.

Definition 5 (Hildebrand and Sanderson [6], Whyburn [12]). A space X is *locally cohesive* if for each $x \in X$ and each neighborhood U of x there exists a region R (connected and open set) such that $x \in R \subseteq \bar{R} \subseteq U$, $\mathrm{bd}(R)$ is connected and $\bar{R}$ is unicoherent mod $\mathrm{bd}(R)$. Such a region R will be called *canonical*.

This definition differs slightly from that of a cohesive space as given in [7] and [12]; however, in our setting of locally compact spaces the definitions are equivalent ([7], p. 73). From [12] we obtain the following

THEOREM 1. *Let $f\colon X \to Y$, where X is locally compact, locally cohesive and metric, and Y is a metric space. Then f is connectivity function if and only if f is a peripherally continuous function.*

Suppose that $f\colon X \to Y$ is a peripherally continuous function, where X is a locally cohesive and locally compact metric space. Let $x \in X$ and let U and V be open sets such that $x \in U$ and $f(x) \in V$. Then there exists a region N, with connected boundary, such that $x \in N \subseteq \bar{N} \subseteq U$ and $f(\mathrm{bd}(N)) \subseteq V$. We verify this as follows (cf. [11]).

Let R be a canonical region about x in U. Then there exists a Q about x in R such that $\bar{Q} \subseteq R$ and $f(\mathrm{bd}(Q)) \subseteq V$. We may assume that Q is connected, for if it is not, then we may consider the component Q' of Q containing x ($\mathrm{bd}(Q') \subset \mathrm{bd}(Q)$, because X is locally connected). Let S be the component of $\bar{R} - \bar{Q}$ that contains $\mathrm{bd}(R)$ and let N be $\bar{R} - S$. It follows that N and $\bar{S}$ are closed connected sets satisfying $\mathrm{bd}(N) \subseteq \mathrm{bd}(Q)$ and $N \cup \bar{S} = \bar{R}$. Thus $N \cap \bar{S} = \mathrm{bd}(N)$ is connected and $f(\mathrm{bd}(N)) \subseteq V$.

A point $x \in X$ is said to be a *local cut point* of X if there exists a connected open set U such that $x \in U$ and $U - x$ is not connected. In the sequel, Z will denote the set of natural numbers.

THEOREM 2. *Let $f\colon X \to Y$ be a connectivity function, where X is a locally cohesive, locally compact, separable metric space without local cut points, and $Y \subseteq L$, where L is a locally convex subset of a normed linear space. Then the function $g\colon X \to L$, where $g(x) = f(x)$ for each $x \in X$, is almost continuous. If Y is a retract of L, then $f\colon X \to Y$ is almost continuous.*

Proof. We may assume by Theorem 1 that g is a peripherally continuous function. Let $B(z, r)$ denote the open ball of radius r centered at z. Suppose that V is an open subset of $X \times L$, with $\Gamma g \subseteq V$. For each $x \in X$ select two numbers $e_1(x)$ and $e_2(x)$ such that $e_1(x) < \operatorname{diam} X/2$, the ball $B\big(f(x), 2e_2(x)\big)$ is convex and

$$B\big(x, e_1(x)\big) \times B\big(f(x), 2e_2(x)\big) \subseteq V.$$

Since X is a locally compact, separable, metric space, there exists a sequence of open sets $B_0, B_1, B_2, \ldots,$ each with compact closure, such that $B_0 = \emptyset$, $\bar{B}_i \subseteq B_{i+1}$ for each i, and

$$\bigcup_{i\in Z} B_i = X.$$

If X is compact, we set $X = B_i$ for $i \geqslant 1$.

For each $x \in X$ there exists an open connected neighborhood $U(x)$ of x such that the following conditions hold:

(1) if $x \in \bar{B}_i - \bar{B}_{i-1}$, then $U(x) \subseteq B_{i+1} - \bar{B}_{i-1}$;

(2) $U(x) \subseteq B\big(x, e_1(x)\big)$ and $f\big(\operatorname{bd}(U(x))\big) \subseteq B\big(f(x), e_2(x)\big)$;

(3) $\operatorname{bd}\big(U(x)\big)$ is connected.

Let $\mathscr{D} = \{U(x) : x \in X\}$ and, for each $i \geqslant 1$, set

$$\mathscr{D}_i = \{U(x) : x \in \bar{B}_i - \bar{B}_{i-1}\}.$$

The collection $\mathscr{D}_1$ covers $\bar{B}_1$, and hence there exists a finite minimal subcover of $\bar{B}_1$. Let the set of members of the finite subcover be denoted by $\mathscr{C}_1$. Suppose that we have a finite minimal subcover $\mathscr{C}_i$ of $\bar{B}_i$. Then $\mathscr{C}_i \cup \mathscr{D}_{i+1}$ is a cover of $\bar{B}_{i+1}$. Let $\mathscr{C}_{i+1}$ denote a finite minimal subcover of $\bar{B}_{i+1}$. Clearly, $\mathscr{C}_i \subseteq \mathscr{C}_{i+1}$. By induction, for each $i \geqslant 1$ there exists a set $\mathscr{C}_i \subseteq \mathscr{D}$ such that $\mathscr{C}_i$ is a finite minimal subcover of $\bar{B}_i$ and $\mathscr{C}_i \subseteq \mathscr{C}_{i+1}$. Also, if $U \in \mathscr{C}_{i+1}$ and $U \cap \bar{B}_i \neq \emptyset$, then $U \in \mathscr{C}_i$. Clearly,

$$\mathscr{C} = \bigcup_{i\in Z} \mathscr{C}_i$$

is a minimal locally finite cover of X.

Let the members of $\mathscr{C}$ be denoted by $U(x_1), U(x_2), U(x_3), \ldots,$ and let $\{p_i : X \to [0, 1] : i \in Z\}$ be a partition of unity subordinate to $\mathscr{C}$. Define $h: X \to L$ by

$$h(x) = \sum_{i\in Z} p_i(x) f(x_i) \Big/ \sum_{i\in Z} p_i(x)$$

which is a continuous function.

Select an $x \in X$. Without loss of generality we may assume that $U(x_1), \ldots, U(x_n)$ are those members of $\mathscr{C}$ that contain x, and that

$$e_2(x_1) = \max\{e_2(x_1), \ldots, e_2(x_n)\}.$$

For each integer i, $1 \leqslant i \leqslant n$, $U(x_1) \cap U(x_i) \neq \emptyset$. Since $\mathscr{C}$ is a minimal cover of X and $\operatorname{diam}(U(x_i)) < \operatorname{diam} X/2$, we have

$$\operatorname{bd}(U(x_1)) \cap \operatorname{bd}(U(x_i)) \neq \emptyset.$$

Let $b \in \operatorname{bd}(U(x_1)) \cap \operatorname{bd}(U(x_i))$. Then

$$d(f(x_1), f(b)) < e_2(x_1) \quad \text{and} \quad d(f(x_i), f(b)) < e_2(x_i) \leqslant e_2(x_1),$$

where d is the metric defined on Y. It follows that

$$d(f(x_1), f(x_i)) < 2e_2(x_1).$$

Thus

$$f(x_1), \ldots, f(x_n) \in B(f(x_1), 2e_2(x_1)) \quad \text{and} \quad h(x) \in B(f(x_1), 2e_2(x_1)).$$

Clearly, $x \in B(x_1, e(x_1))$, and so $(x, h(x)) \in V$. Hence, the map $h: X \to L$ has the property that $\Gamma h \subseteq V$ and, consequently, $g: X \to L$ is almost continuous.

If there exists a retraction $r: L \to Y$, then, by Proposition 1 in [9], $f = r \circ g$ is almost continuous.

COROLLARY 1. *If $f: R \to R$ is a connectivity function, where R is a locally compact, separable, metric absolute neighborhood retract without local separating points, then f is almost continuous. If R has the fixed-point property for continuous functions, then f has a fixed point.*

Proof. By [1], p. 79, there exists a homeomorphism $h: R \to h(R)$, where $h(R)$ is a closed subset of a convex subset C of a normed linear space. Let $R^* = h(R)$. It follows that there exist an open subset X of C and a retraction $r: X \to R^*$. We show that R^* (R) is locally cohesive.

Let $p \in R^*$ and let U be an open neighborhood of p in R^* such that $\overline{U}$ is a Peano continuum. The existence of such a neighborhood follows from [10], p. 22. There exists an open convex neighborhood W of p in X such that $r(W) \subseteq U$. Let V be a neighborhood of p such that $\overline{V} \subseteq W \cap R^* \subseteq U$ and $\overline{V}$ is a Peano continuum. We show that $\overline{V}$ is unicoherent $\operatorname{mod} \operatorname{bd}(V)$. Suppose that there exist closed and connected sets A and B such that $A \cup B = \overline{V}$, $A \subseteq V$ and $A \cap B$ is not connected.

We may assume that A and B are Peano continua. There exist disjoint closed sets C_1 and C_2 such that $A \cap B = C_1 \cup C_2$. There exist an arc $(ax_1b) \subseteq A$ such that $a \in C_1$, $b \in C_2$ and $A \cap B \cap (ax_1b) = \{a, b\}$ and an arc $(ax_2b) \subseteq B$. Then $J = (ax_1b) \cup (ax_2b)$ is a simple closed curve. Let U_1 and U_2 denote open sets such that $C_1 \subseteq U_1$, $C_2 \subseteq U_2$ and $U_1 \cap U_2 = \emptyset$, and let

$$L = A \cap (ax_2b) - \{a, b\}.$$

There exists an open set T such that

$$L \subseteq T, \quad \overline{T} \cap (ax_1b) \subseteq \{a, b\} \quad \text{and} \quad \overline{T} \subseteq U_1 \cup U_2.$$

Set

$$H = A - T \quad \text{and} \quad K = B \cup \bar{T} \cup \{x: x \in \bar{U} - V\}.$$

Then H and K are closed sets such that

$$H \cup K = \bar{U}, \quad H \cap J = (ax_1b), \quad K \cap J = (ax_2b)$$

and

$$\{a, b\} \subseteq H \cap K \subseteq U_1 \cup U_2.$$

For $i = 1, 2$, we define the function

$$r_i: H \cap K \cup (ax_ib) \to (ax_ib)$$

by

$$r_i(x) = \begin{cases} a & \text{if } x \in H \cap K \cap U_1, \\ b & \text{if } x \in H \cap K \cap U_2, \\ x & \text{if } x \in (ax_ib). \end{cases}$$

Clearly, r_i is continuous. Tietze's extension theorem implies that r_1 can be extended to a continuous map $s_1: H \to (ax_1b)$ and r_2 can be extended to a continuous map $s_2: K \to (ax_2b)$. Then the continuous map $s: \bar{U} \to J$, where

$$s(x) = \begin{cases} s_1(x) & \text{if } x \in H, \\ s_2(x) & \text{if } x \in K, \end{cases}$$

is a retraction of $\bar{U}$ onto J and $s \circ r: W \to J$ is a retraction of W onto J. A simple homology argument, e.g., shows that this is impossible. Thus we conclude that $\bar{V}$ is unicoherent $\operatorname{mod} \operatorname{bd}(V)$.

Since $\bar{V} - p$ is connected, there exists an open connected set Q such that $p \in Q \subseteq \bar{Q} \subseteq V$ and $\bar{V} - Q$ is connected ([10], p. 50). Since $\bar{V}$ is unicoherent $\operatorname{mod} \operatorname{bd}(V)$, Q is unicoherent $\operatorname{mod} \operatorname{bd}(Q)$. Set $C = \bar{Q}$ and $D = \bar{V} - Q$. Then C and D are closed connected sets, $C \cup D = \bar{V}$ and $C \cap D = \operatorname{bd}(Q)$ which, consequently, must be connected. Thus Q satisfies the local cohesive condition and we conclude that R^* (R) is locally cohesive.

By [6], p. 238, $h \circ f: R \to R^*$ is a connectivity function and, by Theorem 2, $h \circ f$ is almost continuous. By [9], p. 260, $f = h^{-1} \circ (h \circ f)$ is almost continuous. This together with [9], p. 252, completes the proof.

Finally, we establish an extension to connectivity maps of the Schauder fixed-point theorem.

COROLLARY 2. *Let $f: X \to X$ be a connectivity function and X a compact convex subset of a normed linear space. Then f has a fixed point.*

Proof. It is easy to see that if X has a local cut point, then X is an interval. In this case the result follows from [5]. If X does not have local cut points, then the result follows from Corollary 1.

THEOREM 3 (generalized Schauder fixed-point theorem). *If $f: X \to X$ is a connectivity function, X a closed convex subset of a Banach space, and $\overline{f(X)}$ compact, then f has a fixed point.*

Proof. Let C be the closure of the convex hull of $f(X)$. Then C is a compact convex set and the result follows from Corollary 2 by restricting f to C.

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