COLLOQUIUM MATHEMATICUM

VOL. LX/LXI DÉDIÉ À M. ANTONI ZYGMUND 1990

THE BLOCK DECOMPOSITION OF THE WEIGHTED BESOV SPACES

BY

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We introduce the block space A_ρ defined by

$$A_\rho = \left\{ f : [0, 2\pi] \to \mathbf{R};\ f(t) = \sum_{n=0}^{\infty} c_n a_n(t);\ \sum_{n=0}^{\infty} |c_n| < \infty \right\}.$$

Each a_n is a *weighted block*, that is, a real-valued function a, defined on $[0, 2\pi]$, which is $a(t) = \frac{1}{\rho(|I|)} \chi_I(t)$, where I is an interval and $|I|$ is the length of I, χ_I the characteristic function of I and ρ is a non-negative continuous function defined on $[0, \infty)$ such that

(i) $\rho(0) = 0$,
(ii) ρ is increasing,
(iii) $\rho(t)/t$ is integrable on $[0, 2\pi]$,
(iv) $\int_0^h (\rho(t)/t)\,dt \le C\rho(h)$ and
(v) $\int_h^{2\pi} (\rho(t)/t^2)\,dt < C\rho(h)/h$.

C is an absolute constant and may not be the same at every occurrence, throughout this paper.

We endow A_ρ with the norm $\|f\|_{A_\rho} = \inf \sum_{n=0}^{\infty} |c_n|$, where the infimun is taken over all possible representations of f.

For a similar definition of A_ρ see [4]. In general, for the block functions in the unweighted case see for example [12].

We define the weighted Besov spaces:

$$\Lambda_\rho = \left\{ f : [0, 2\pi] \to \mathbf{R};\ \|f\|_{\Lambda_\rho} = \int_0^{2\pi} \int_0^{2\pi} \frac{|f(x) - f(y)|}{|x - y|^2} \rho(x - y)\,dx\,dy < \infty \right\}$$

where ρ satisfies the earlier conditions.

These spaces have been studied in [6]–[9]; for the unweighted case see [2], [5], [10] and [11].

1980 *Mathematics Subject Classification*: Primary 42A99.

Key words and phrases: Besov, Lipschitz spaces, equivalence of Banach spaces, weighted spaces, analytic functions, block decomposition.

Notice that for $1 < p < \infty$ and $\rho(t) = t^{1/p}$, Λ_ρ is the space denoted by $\Lambda(1-1/p, 1, 1)$ and extensively studied by M. Taibleson [11], T. M. Flett [5] and others.

The main purpose of this paper is to show that the spaces A_ρ and Λ_ρ are equivalent as Banach spaces and also to give an analytic characterization of these spaces. We shall do this by showing a series of results.

THEOREM A. *If $f \in A_\rho$ then $f \in \Lambda_\rho$. Moreover, $\|f\|_{\Lambda_\rho} \le C\|f\|_{A_\rho}$ where C is an absolute constant.*

Proof. We just need to prove the result for $f_h(x) = \frac{1}{\rho(|I|)}\chi_I(t)$, where $I = [0, h]$, since A_ρ is invariant under translations.

Therefore, all we need to prove is that

$$\int_0^{2\pi}\int_0^{2\pi} \frac{|f_h(x) - f_h(y)|}{|x-y|^2}\rho(x-y)\,dx\,dy \le C\,,$$

where C is an absolute constant.

In fact,

$$\|f\|_{\Lambda_\rho} \le \frac{1}{\rho(h)}\left[\int_0^h\int_h^{2\pi} \frac{\rho(x-y)}{|x-y|^2}\,dx\,dy + \int_h^{2\pi}\int_0^h \frac{\rho(x-y)}{|x-y|^2}\,dx\,dy\right] = A + B\,,$$

$$A = \frac{1}{\rho(h)}\int_0^h\int_h^{2\pi} \frac{\rho(x-y)}{|x-y|^2}\,dx\,dy \le \frac{2}{\rho(h)}\int_0^h \frac{\rho(t)}{t}\,dt + \frac{2h}{\rho(h)}\int_h^{2\pi}\frac{\rho(t)}{t^2}\,dt\,.$$

Now, using properties (iv) and (v) of ρ we get $A \le C$. The estimate for B is similar, so that we have $B \le C$.

Our argument shows that, for f_h as above, we have $\|f_h\|_{\Lambda_\rho} \le C$, and consequently, if $f \in A_\rho$ it follows that $\|f\|_{\Lambda_\rho} \le C\|f\|_{A_\rho}$. So Theorem A is proved.

We now introduce the space S_ρ of all those analytic functions F on the unit disc $\mathbf{D}$ for which

$$\|F\|_{S_\rho} = \int_0^1\int_{-\pi}^{\pi} |F'(re^{i\theta})|\frac{\rho(1-r)}{1-r}\,d\theta\,dr < \infty\,,$$

where ρ satisfies conditions (i)–(v) above beside of

$$(*) \qquad \int_h^{2\pi}\frac{\rho(t)}{t^3}\,dt \le C\frac{\rho(h)}{h^2}$$

where C is an absolute constant. The dash means the derivative of F with respect to z. For this space S_ρ the reader is referred to [4].

Notice that (v) in the definition of ρ implies $(*)$ but not conversely. For example, $\rho(t) = t$ or $\rho(t) = \ln(1+t)$ satisfy $(*)$ but not (v).

THEOREM B. *If $f \in \Lambda_\rho$ then $F \in S_\rho$ where*

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} f(e^{it})\, dt .$$

Moreover, $\|F\|_{S_\rho} \le C \|f\|_{\Lambda_\rho}$, where C is an absolute constant.

Before proving this theorem we need the following result.

LEMMA C. *For $0 < |t| < \pi$, we have*

$$\int_0^1 \frac{1}{1-2r\cos t + r^2} \cdot \frac{\rho(1-r)}{1-r}\, dr \le C \frac{\rho(t)}{t^2},$$

where C is an absolute constant.

Proof. Let

$$I(t) = \int_0^1 \frac{1}{1-2r\cos t + r^2} \cdot \frac{\rho(1-r)}{1-r}\, dr .$$

Notice that

1) $\dfrac{1}{1-2r\cos t + r^2} \le \dfrac{2}{(1-r)^2}$ for $0 < r < 1$ and $0 < |t| < \pi$,

2) $\dfrac{1}{1-2r\cos t + r^2} \le \dfrac{C}{t^2}$, C an absolute constant.

For these estimates see [13, p. 97].

For $0 < t < 1$, we have

$$I(t) = \left(\int_0^{1-t} + \int_{1-t}^1 \right) \frac{1}{1-2r\cos t + r^2} \cdot \frac{\rho(1-r)}{1-r}\, dr = A + B .$$

Using 1) above and condition (v) on ρ we get

$$A \le C \int_0^{1-t} \frac{\rho(1-r)}{(1-r)^3}\, dr \le C \int_t^1 \frac{\rho(s)}{s^3}\, ds \le \frac{C}{t} \int_t^{2\pi} \frac{\rho(s)}{s^2}\, ds \le C \frac{\rho(t)}{t^2} .$$

Using 2) and (iv) we have

$$B \le \frac{C}{t^2} \int_{1-t}^1 \frac{\rho(1-r)}{1-r}\, dr \le \frac{C}{t^2} \int_0^t \frac{\rho(s)}{s}\, ds \le C \frac{\rho(t)}{t^2} .$$

Consequently, $I(t) \le C\rho(t)/t^2$ where C is an absolute constant.

For $1 \le t \le \pi$, we use (iv), (ii) and 2) to get

$$I(t) \le \frac{C}{t^2} \int_0^1 \frac{\rho(1-r)}{1-r}\, dr = \frac{C}{t^2} \int_0^1 \frac{\rho(s)}{s}\, ds \le C\frac{\rho(1)}{t^2} \le C\frac{\rho(t)}{t^2}.$$

The case $-\pi < t < 0$ is similar, therefore Lemma C is proved.

Proof of Theorem B. We have

$$F'(z) = \frac{1}{\pi} \int_0^{2\pi} \frac{[f(e^{it}) - f(e^{i\theta})]e^{it}}{(e^{it} - z)^2}\, dt,$$

and so

$$|F'(z)| \le \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{|f(e^{i(t+\theta)}) - f(e^{i\theta})|}{1 - 2r\cos t + r^2}\, dt.$$

Let us now estimate $\|F\|_{S_\rho}$. We have

$$\begin{aligned}\|F\|_{S_\rho} &< \frac{1}{\pi} \int_0^1 \int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} \frac{|f(e^{i(t+\theta)}) - f(e^{i\theta})|}{1 - 2r\cos t + r^2}\, dt \right) \frac{\rho(1-r)}{1-r}\, d\theta\, dr \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(e^{i(t+\theta)}) - f(e^{i\theta})| \\ &\qquad \times \left(\int_0^1 \frac{1}{1 - 2r\cos t + r^2} \cdot \frac{\rho(1-r)}{1-r}\, dr \right) d\theta\, dt.\end{aligned}$$

Using Lemma C we get

$$\|F\|_{S_\rho} \le \frac{C}{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(e^{i(t+\theta)}) - f(e^{i\theta})|}{t^2} \rho(t)\, d\theta\, dr,$$

that is, $\|F\|_{S_\rho} \le C\|f\|_{\Lambda_\rho}$, where C is an absolute constant. So Theorem B is proved.

LEMMA D. *If f is in* Lip ρ, *that is*, $|f(x+h) - f(x)| \le M\rho(h)$, *then*

$$|F'(z)| \le C\frac{\rho(1-r)}{1-r} \quad \textit{where} \quad F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} f(e^{it})\, dt, \quad z = re^{i\theta},$$

and C is an absolute constant.

Proof. Usually the least constant M in the definition of Lip ρ is denoted by $\|f\|_{\mathrm{Lip}\,\rho}$.

Arguing as in the proof of Theorem B, and using the fact that

$$\frac{1}{1 - 2r\cos t + r^2} \le \frac{C}{(1-r)^2 + t^2}$$

where C is an absolute constant, $0 < r < 1$ and $0 < |t| \leq \pi$ (see [13, p. 96]), we get

$$|F'(z)| \leq \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{|f(e^{i(t+\theta)}) - f(e^{i\theta})|}{1 - 2r\cos t + r^2}\, dt \leq C \int_{-\pi}^{\pi} \frac{\rho(|t|)}{(1-r)^2 + t^2}\, dt\,.$$

Taking $s = t/(1-r)$ we get

$$(**) \qquad |F'(z)| \leq \frac{C}{1-r} \int_0^{\pi/(1-r)} \frac{\rho(s(1-r))}{1+s^2}\, ds\,.$$

Let us estimate this last integral:

$$\int_0^{\pi/(1-r)} \frac{\rho(s(1-r))}{1+s^2}\, ds = \left(\int_0^1 + \int_1^{\pi/(1-r)} \right) \frac{\rho(s(1-r))}{1+s^2}\, ds = A + B\,.$$

Using (ii), we have $\rho(s(1-r)) \leq \rho(1-r)$ and so $A \leq \frac{\pi}{4}\rho(1-r)$. Next,

$$\begin{aligned} B \;&\leq \int_1^{\pi/(1-r)} \frac{\rho(s(1-r))}{s^2}\, ds = (1-r)^2 \int_1^{\pi/(1-r)} \frac{\rho(s(1-r))}{(s(1-r))^2}\, ds \\ &= (1-r) \int_{1-r}^{\pi} \frac{\rho(t)}{t^2}\, dt\,. \end{aligned}$$

Therefore using (v) we get $B \leq C\rho(1-r)$.

Putting the estimates for A and B in (**) we get $|F'(z)| \leq C\frac{\rho(1-r)}{1-r}$, with $z = re^{i\theta}$ and C an absolute constant. So Lemma D is proved.

Let $F(z) = \sum_{n=0}^{\infty} b_n z^n$ be the analytic extension of a function f in Λ_ρ, and let g in $\operatorname{Lip}\rho$ have analytic extension $G(z) = \sum_{n=0}^{\infty} a_n z^n$.

Define a linear functional on Λ_ρ by

$$\psi_g(f) = \lim_{r\to 1} \int_0^{2\pi} F(re^{i\theta}) G'(re^{-i\theta})\, d\theta\,.$$

We claim that ψ_g belongs to Λ_ρ^* (the dual of Λ_ρ). In fact, one can easily see that

$$\begin{aligned} &\lim_{r\to 1} \int_0^{2\pi} F(re^{i\theta}) G'(re^{-i\theta})\, d\theta \\ &\qquad = F(0)G'(0) + 2\int_0^1 \int_0^{2\pi} F'(re^{i\theta}) G'(re^{-i\theta}) e^{i\theta}\, d\theta\, dr\,. \end{aligned}$$

Now as Λ_ρ is continuously contained in S_ρ by Theorem B and $|G'(re^{-i\theta})| \le C\|g\|_{\mathrm{Lip}\,\rho}\rho(1-r)/(1-r)$ by Lemma D, we get

$$|\psi_g(f)| \le C\|F\|_{S_\rho}\|g\|_{\mathrm{Lip}\,\rho} \le C\|g\|_{\mathrm{Lip}\,\rho}\|f\|_{\Lambda_\rho},$$

that is, ψ_g is a bounded linear functional on Λ_ρ.

This last inequality, which is some sort of Hölder's inequality, tells us that, for any fixed g in $\mathrm{Lip}\,\rho$, ψ_g is a bounded linear functional on Λ_ρ, that is, $\mathrm{Lip}'\,\rho$ is continuously contained in Λ_ρ^*, written $\mathrm{Lip}'\,\rho \subseteq \Lambda_\rho^*$, where $\mathrm{Lip}'\,\rho = \{g'; g \in \mathrm{Lip}\,\rho\}$ and the dash means derivative.

One can easily see that the dual space of A_ρ is $\mathrm{Lip}'\,\rho$, see Theorem 4.1 in [4, p. 188]. Therefore as $A_\rho \subseteq \Lambda_\rho$ we have $\Lambda_\rho^* \subseteq \mathrm{Lip}'\,\rho$. Thus as $A_\rho \subseteq \Lambda_\rho \subseteq S_\rho$ we have the following duality theorem.

THEOREM E (Duality Theorem). *The dual spaces of A_ρ, Λ_ρ and S_ρ are all equivalent to* $\mathrm{Lip}'\,\rho$, *i.e.* $A_\rho^* \cong \Lambda_\rho^* \cong S_\rho^* \cong \mathrm{Lip}'\,\rho$. *Moreover, the duality pairing is given by the functional ψ_g.*

It is not difficult to use the Hahn–Banach Theorem to show that A_ρ is dense as a subset of both Λ_ρ and S_ρ, therefore we have the following situation: A_ρ, Λ_ρ and S_ρ have the same duals, moreover the inclusion mappings $I : A_\rho \hookrightarrow \Lambda_\rho$ and $I : A_\rho \hookrightarrow S_\rho$ are dense and continuous. It follows that A_ρ, Λ_ρ and S_ρ are equivalent as Banach spaces. Thus we have proved the main result in this paper:

THEOREM F. *Let $\rho : [0,\infty) \to [0,\infty)$ be a continuous function satisfying conditions* (i)–(v). *Then*

$$f \in A_\rho \Leftrightarrow f \in \Lambda_\rho \Leftrightarrow F \in S_\rho \quad \textit{where} \quad F(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} f(e^{it})\,dt.$$

Moreover, the norms $\|f\|_{A_\rho}$, $\|f\|_{\Lambda_\rho}$ and $\|F\|_{S_\rho}$ are equivalent.

Comments. The space A_ρ is called the *block decomposition* of the weighted Besov space Λ_ρ or S_ρ.

For ρ as in Theorem F, we obtain Theorem 9.2, p. 207 in [4] with very simple calculations.

For $\rho(t) = t^{1/p}$, $1 < p < \infty$ we obtain the main result in [2], which is Theorem C, p. 684.

We would like to thank Professors Jack Brown and Gary Sampson with whom we had some useful conversations.

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Reçu par la Rédaction le 20.2.1990