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COPRODUCTS OF BOOLEAN ALGEBRAS AND CHAINS WITH APPLICATIONS TO POST ALGEBRAS

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1. Introduction. Let $\mathscr{C}$ be the category of finite chains with $0 < 1$ and isotone maps which preserve $0, 1$; and let $\mathscr{B}$ be the category of non-degenerate Boolean algebras and Boolean homomorphisms. One of the objectives of this paper* is to prove that the category $\mathscr{P}$ of Post algebras and Post homomorphisms is equivalent with $\mathscr{B} \times \mathscr{C}$ (Theorem 2.2). As an application of this result we will exhibit the injectives and projectives in $\mathscr{P}$, and also the essential injective extensions of objects in $\mathscr{P}$. As we will see the concept of a Post algebra leads to a study of the coproduct of a Boolean algebra B and a chain C in the category of distributive lattices with $0, 1$. These coproducts could be considered as a generalization of Post algebras (see also Chang and Horn [3], Traczyk [11], Dwinger [5]). It will be shown (Theorem 3.2) that such a coproduct is isomorphic with a subdirect of C^S for a suitable set S. In fact, this subdirect product is characterized by being the lattice generated by the set of diagonal elements in C^S and a Boolean sublattice of C^S. This result will, in particular, yield a new structure theorem for Post algebras. It also provides an easy characterization of Post algebras whose center is a complete field of sets.

It was proven (Dwinger [4], also see Rousseau [10]) that for every Post algebra P the distinguished chain of constants is uniquely determined. The last section deals with the question of whether this uniqueness theorem also holds if the distinguished finite chain is replaced by an infinite chain, i.e., for coproducts $B * C$, where C is infinite. We will answer this question in the negative but we will present a result which improves the uniqueness theorem for Post algebras.

We recall some definitions.

Definition 1.1. A *Post algebra* P is a distributive lattice with $0, 1$ which contains a finite chain $C = \{e_0, \ldots, e_{n-1}\}$, where $0 = e_0 < e_1 < \ldots$

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$< e_{n-1} = 1$, $n \geqslant 2$, and such that each $x \in P$ can be uniquely represented in the form $x = \sum_{i=1}^{n-1} a_i e_i$, where $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_{n-1}$ and $a_1, \ldots, a_{n-1}$ are complemented in P. Such a representation is called *monotonic*.

The uniqueness theorem, to which we referred above, states that if P is a Post algebra, then there is exactly one chain in P satisfying the conditions on C in Definition 1.1. Thus, we refer to C as the *chain of constants in P* and say that P is a Post algebra of order n, where $|C| = n$. The set of complemented elements of P will be referred to as the center of P. More generally, the Boolean sublattice of all complemented elements of a distributive lattice with $0 < 1$ is called the *center* of L.

Since most of our work will be in the category $\mathscr{D}$ of distributive lattices with $0 < 1$ and lattice homomorphisms which preserve $0, 1$, we will refer to objects and morphisms in $\mathscr{D}$ as lattices and homomorphisms. Thus homomorphisms will always preserve $0, 1$ and sublattices will mean $(0, 1)$-sublattices, etc. We stipulate that all chains be in $\mathscr{D}$. The coproduct ($=$ free product) of any two objects L_1 and L_2 in $\mathscr{D}$ exists and will be denoted by $L_1 * L_2$. The corresponding homomorphisms of L_1 and L_2 in $L_1 * L_2$ are injections. We will often identify L_1 and L_2 with their images in $L_1 * L_2$. Thus, if we write $L = L_1 * L_2$ without specifying what the injections are, we will simply mean that L_1 and L_2 are sublattices of L and that the corresponding injections are the inclusions $L_i \to L_1 * L_2$, $i = 1, 2$. It is therefore also clear what is meant by the statement that L is the coproduct of sublattices L_1 and L_2. It is known (Holsztyński, cf. [10], p. 134; also Grätzer and Lakser [8]) that $L = L_1 * L_2$ if and only if $L_1 \cup L_2$ generates L and for $a_1, b_1 \in L_1$ and $a_2, b_2 \in L_2$

(1) $$a_1 a_2 \leqslant b_1 + b_2 \text{ implies } a_1 \leqslant b_1 \text{ or } a_2 \leqslant b_2.$$

Finally, whenever we talk about coproducts in the sequel we will mean coproducts in $\mathscr{D}$.

It was proven in Rousseau [10] that if P is a Post algebra with center B and chain of constants C then $P = B * C$. Conversely, if B is a Boolean algebra and C a finite chain then $B * C$ is a Post algebra in which C is the chain of constants in $B * C$. It therefore follows from (1) that $L \in \mathrm{Obj}.\mathscr{D}$ is a Post algebra if and only if L is generated by the union of its center B and a finite subchain that satisfies

(1′) $$a \in B;\, c, d \in C,\, ac \leqslant d \text{ implies } a = 0 \text{ or } c \leqslant d.$$

Definition 1.2. Let P and P_1 be Post algebras with chains of constants C_1 and C_2, respectively. A *Post homomorphism* $h\colon P \to P_1$ is a lattice homomorphism such that $h(C) \subseteq C_1$.

Post homomorphisms can be realized as the homomorphisms associated with a certain class of structures (see Grätzer [7]) of type

$\langle\langle 2, 2, 0, 0\rangle, \langle 1\rangle\rangle$. Indeed, a Post algebra P with chain of constants $C = \{e_0, \ldots, e_{n-1}\}$ is a structure $\langle L; \{+, \cdot, 0, 1\}, \{R\}\rangle$, where $\langle L; +, \cdot, 0, 1\rangle$ is a lattice, $R = \{(e_i) | \; i = 0, \ldots, n-1\}$ and each $x \in L$ has a monotonic representation $\sum_{i=1}^{n-1} a_i e_i$; $a_1, \ldots, a_n$ in the center of L. As we shall see in the next section, however, it appears to be more useful to consider Post algebras and homomorphisms from a categorical point of view.

2. The category of Post algebras. We first prove a lemma (cf. Corollary 3.3).

LEMMA 2.1. *Let $B \in \mathrm{Obj}.\mathscr{B}$ and $C \in \mathrm{Obj}.\mathscr{C}$. Then B is the center of $B * C$.*

Proof. Let B' be the center of $B * C$. Clearly, $B \subseteq B'$. For the converse inclusion, suppose $C = \{e_0, \ldots, e_{n-1}\}$, where $0 = e_0 < e_1 < \ldots < e_{n-1} = 1$ and $a \in B'$. Now a and its complement $\bar{a}$ can be written in the form $a = \sum_{i=1}^{n-1} a_i e_i$ and $b = \sum_{j=1}^{n-1} b_j e_j$, respectively, where $a_i, b_j \in B$.

For each $i \in \{1, \ldots, n-1\}$,

$$0 = ab \geqslant (a_i e_i)(b_{n-1} e_{n-1}) = a_i b_{n-1} e_i.$$

So, by (1′),

$$a_i b_{n-1} = 0 \quad \text{for } i = 1, \ldots, n-1. \tag{2}$$

Next,

$$1 = a+b \leqslant \Big(\sum_{i=1}^{n-2} e_i\Big) + a_{n-1} + \Big(\sum_{j=1}^{n-2} e_i\Big) + b_{n-1} = e_{n-2} + a_{n-1} + b_{n-1},$$

which implies

$$a_{n-1} + b_{n-1} = 1. \tag{3}$$

Combining (2) and (3), we have

$$a_i e_i \leqslant \bar{b}_{n-1} e_{n-1} = a_{n-1} \qquad (i = 1, \ldots, n-1)$$

and so

$$a = \sum_{i=1}^{n-1} a_i e_i = a_{n-1}.$$

Hence $a \in B$.

THEOREM 2.2. *The categories $\mathscr{P}$ and $\mathscr{B} \times \mathscr{C}$ are equivalent.*

Proof. We define (covariant) functors $\Phi\colon \mathscr{P} \to \mathscr{B} \times \mathscr{C}$ and $\Psi\colon \mathscr{B} \times \mathscr{C} \to \mathscr{P}$ as follows. For every $P \in \mathrm{Obj}.\mathscr{P}$ let $\Phi(P) = (B, C)$, where B is the center of P and C is the chain of constants in P. If $h\colon P \to P_1$ is a morphism of $\mathscr{P}$, $\Phi(P) = (B, C)$, $\Phi(P_1) = (B_1, C_1)$, define $\Phi(h) = (h_1, h_2)$, where h_1 and h_2 are the restrictions of h to B and C, respectively. It is obvious that Φ is a functor. Next, let $(B, C) \in \mathrm{Obj.}\, \mathscr{B} \times \mathscr{C}$. Let $\Psi(B, C) = B * C$, where, in accordance with our convention, $B * C$ is the coproduct of the sublattices B and C and where the corresponding injections are the inclusion maps. Then $\Psi(B, C)$ is a Post algebra. Let $h = (h_1, h_2)$: $(B, C) \to (B_1, C_1)$ be a morphism in $\mathscr{B} \times \mathscr{C}$. Let $\Psi(h)$: $\Psi(B, C) \to \Psi(B_1, C_1)$ be the uniquely determined Post homomorphism such that $\Psi(h)|B = h_1$ and $\Psi(h)|C = h_2$. Again it is obvious that Ψ is a functor. Now let h: $P \to P_1$ be a morphism of $\mathscr{P}$, $\Phi(P) = (B, C)$, $\Phi(P_1) = (B_1, C_1)$. Then it follows immediately from equalities $P = B * C$ and $P_1 = B_1 * C_1$ that $\Psi(\Phi(P)) = P$ and $\Psi(\Phi(P_1)) = P_1$. Moreover, $\Phi(h) = (h_1, h_2)$ and hence $\Psi(\Phi(h)) = h$. Finally, let $h = (h_1, h_2)$: $(B, C) \to (B_1, C_1)$ be a morphism in $\mathscr{B} \times \mathscr{C}$. It follows from Lemma 3 that $\Phi(\Psi(B, C)) = (B, C)$ and $\Phi(\Psi(B_1, C_1)) = (B_1, C_1)$ and then from the definition of Φ and Ψ we have $\Phi(\Psi(h)) = h$. This concludes the proof of the theorem.

It is now easy to prove

THEOREM 2.3. *A Post homomorphism is monic* (*epic*) *if and only if it is one to one* (*onto*).

Proof. Let B_i and C_i be the center and chain of constants of P_i, $i = 1, 2$, and $h_1 = h|B_1$, $h_2 = h|C_1$. If h is monic (epic), then (h_1, h_2) is monic (epic) by Theorem 4, and so h_1 is monic (epic) in $\mathscr{B}$ and h_2 is monic (epic) in $\mathscr{C}$. But it is known that in $\mathscr{B}$ and $\mathscr{C}$ monic and epic coincide with 'one to one' and 'onto', respectively. Now if h_1 and h_2 are onto then, since $h_1(B_1) \cup h_2(C_1) = B_2 \cup C_2$ generates P_2, h is onto. On the other hand suppose h_1 and h_2 are one to one,

$$x = \sum_{i=1}^{n-1} a_i e_i \quad \text{and} \quad y = \sum_{i=1}^{n-1} b_i e_i$$

are monotonic representations and

$$\sum_{i=1}^{n-1} h_1(a_i) h_2(e_i) = \sum_{i=1}^{n-1} h_1(b_i) h_2(e_i).$$

Since the same members of C_2 appear and are strictly increasing on both sides of the expression, it is easily seen from Definition 1.1 that $h_1(a_i) = h_1(b_i)$ for $i = 1, \ldots, n$. Since h_1 is one to one it follows that h is one to one. The converse is immediate.

LEMMA 2.4. *Every object in $\mathscr{C}$ is both projective and injective.*

Proof. Let $C, C_1, C_2 \in \mathrm{Obj.}\, \mathscr{C}$. To show C is injective let h: $C_1 \to C$ be a morphism and g: $C_1 \to C_2$ a mononmorphism. Then g is one to one and the required morphism from C_2 to C is defined by

$$x \to \max\{h(u) |\, u \in C_1,\, g(u) \leqslant x\} \quad \text{for each } x \in C_2.$$

Next, suppose $h\colon C \to C_1$ is a morphism and $f\colon C_2 \to C_1$ is epic. The morphism from C to C_2, which is required to show that C is projective, is given by

$$x \to \max\{u \in C_2 | f(u) \leqslant h(x)\} \quad \text{for } x \in C.$$

Let P be a Post algebra with center B and chain of constants C. Because of Theorem 2.2, P is injective (projective) if and only if B is injective (projective) in $\mathscr{B}$ and C is injective (projective) in $\mathscr{C}$. Now the injectives in B are exactly the complete Boolean algebras, where as the best theorem that exists currently about projectives in $\mathscr{B}$ is that they include all countable Boolean algebras. Thus:

THEOREM 2.5. *Either of the following two conditions are necessary and sufficient for a Post algebra P to be injective:*

(i) *P is complete.*

(ii) *The center of P is complete.*

If P is countable, then it is projective.

Proof. The proof is immediate from the above remarks and the fact that (i) and (ii) are equivalent [6].

We recall the following definitions (cf. [2]). The setting is an arbitrary category $\mathscr{A}$.

Definition 2.6. A monomorphism $f\colon A \to B$ is *essential*, and B is an *essential extension* of A, provided that each morphism $g\colon B \to C$ with the property that gf is a monomorphism is itself a monomorphism. If f is not an isomorphism, then B is a *proper essential extension.*

Using straightforward categorical arguments, it is easy to prove that for categories $\mathscr{A}_1$ and $\mathscr{A}_2$ a morphism $h = (h_1, h_2)\colon (A_1, A_2) \to (A_1', A_2')$ in $\mathscr{A}_1 \times \mathscr{A}_2$ is essential if and only if $h_1\colon A_1 \to A_1'$ and $h_2\colon A_2 \to A_2'$ are essential.

THEOREM 2.7. *For Post algebras P_1 and P_2, the following are equivalent.*

(i) *P_2 is an injective essential extension of P_1.*

(ii) *The orders of P_1 and P_2 are equal and the center of P_2 is a normal completion of the center of P_1.*

(iii) *P_2 is a normal completion of P_1.*

Proof. Let B_i be the center of P_i and let C_i be the chain of constants in P_i for $i = 1, 2$. Then, by our previous remarks and Theorem 2.2, (i) holds if and only if B_2 is an injective essential extension of B_1 (in $\mathscr{B}$) and C_2 is an injective essential extension of C_1 (in $\mathscr{C}$). But a finite chain has no proper such extensions and the injective essential extensions of Boolean algebras are their normal completions [1]. Thus, (i) and (ii) are equivalent. The equivalence of (ii) and (iii) follows immediately from Theorem 2.6 in [6] and the uniqueness of the normal completion.

3. Coproducts of Boolean algebras and chains. Since Post algebras are the coproducts of a Boolean algebra with a finite chain, it seems natural to investigate the structure of coproducts of Boolean algebras with arbitrary chains (in $\mathscr{D}$). The relation that exists between Post algebras and coproducts of a Boolean algebra and a finite chain is reflected in the property that every element of such a coproduct has a unique representation in the sense of Definition 1.1 (Rousseau [10]). This is a special case of the following theorem

THEOREM 3.1. *A lattice L is the coproduct of sublattices B and C, where B is a Boolean algebra and C is a chain, if and only if each $x \in L$ can be uniquely represented in the form*

$$x = \sum_{i=1}^{n} a_i e_i, \qquad n \geqslant 1, \tag{4}$$

where $\{e_1, \ldots, e_n\} \subseteq C$, $\{a_1, \ldots, a_n\} \subseteq B$ *and* $0 = e_1 < \ldots < e_n$, $1 = a_1$, $a_2 > \ldots > a_n > 0$.

Proof. For the sufficiency we need only prove (1′). Thus, suppose $ae \leqslant f$, where $a \in B$, $\{e, f\} \subseteq C$ and $f < e$. Then $1 \cdot 0 + a \cdot e = 1 \cdot 0 + af$, $0 < e$, so if $0 < f$, then the unique representation hypothesis implies $e = f$, a contradiction. But $f = 0$ implies $1 \cdot 0 + ae = 1 \cdot 0 + 0e$, and hence $a = 0$.

For the necessity, we suppose $L = B * C$. If $x = 0$, we have the representation $x = 1 \cdot 0$. Now if $x \neq 0$, then x can be written as $x = \sum_{i=1}^{n} b_i f_i$, where $b_i \in B$, $f_i \in C$. Since C is a chain, we can assume $f_1 < \ldots < f_n$. Also we can assume $b_1 \geqslant \ldots \geqslant b_n$ (cf. [11], p. 194). Now by dropping all but the maximal members of $\{b_i f_i \mid i = 1, \ldots, n\}$, we see that x can be written as

$$x = \sum_{j=1}^{m} b_{i_j} f_{i_j}, \qquad \text{where } 0 < f_{i_1} < \ldots < f_{i_m} \text{ and } b_{i_1} > \ldots > b_{i_m}.$$

The required representation for x is therefore

$$x = 1 \cdot 0 + \sum_{j=1}^{m} b_{i_j} f_{i_j}.$$

Finally, for uniqueness, let

$$x = \sum_{i=1}^{n} a_i e_i = \sum_{j=1}^{m} b_j f_j,$$

where $n \leqslant m$, $0 = e_1 < \ldots < e_n$, $0 = f_1 < \ldots < f_m$, $1 = a_1$, $a_2 > \ldots > a_n > 0$ and $1 = b_1$, $b_2 > \ldots > b_n > 0$. Then $a_1 = b_1 = 1$ and $e_1 = f_1 = 0$.

Suppose $a_i = b_i$, $e_i = f_i$ for $1 \leqslant i \leqslant k$ and $k < n$. Then

$$a_{k+1}e_{k+1} \leqslant \sum_{i=1}^{k} f_i + \sum_{i=k+1}^{m} b_i = e_k + b_{k+1},$$

so, by (1), $a_{k+1} \leqslant b_{k+1}$. Similarly $b_{k+1} \leqslant a_{k+1}$. Now

$$a_{k+1}e_{k+1} \leqslant \sum_{i=1}^{k+1} f_i + \sum_{i=k+2}^{m} b_i.$$

If $k+1 = m$, then this last inequality reads $a_{k+1}e_{k+1} \leqslant f_{k+1}$, so $e_{k+1} \leqslant f_{k+1}$; if $k+1 < m$, then we have $a_{k+1}e_{k+1} \leqslant f_{k+1} + b_{k+2}$. But $a_{k+1} \nleqslant b_{k+2}$ or else $b_{k+1} = a_{k+1} \leqslant b_{k+2}$, a contradiction; hence $e_{k+1} \leqslant f_{k+1}$. Similarly, $f_{k+1} \leqslant e_{k+1}$. Hence $a_i = b_i$ and $e_i = f_i$ for $i = 1, \ldots, n$. Finally, if $m > n$, then

$$b_{n+1}f_{n+1} \leqslant \sum_{i=1}^{n} e_i = e_n, \qquad \text{so} \qquad f_{n+1} \leqslant e_n = f_n,$$

a contradiction; so $m = n$ and the proof is complete.

We will now present our main structure theorem for coproducts of Boolean algebras and chains. First, an element $x \in C^S$, where C is a chain and S is a set, is called a *diagonal element* if $x(s)$ is constant for each $s \in S$.

THEOREM 3.2. *Let B be a Boolean algebra and C a chain. Then $B * C$ is isomorphic with a subdirect product L of C^S for some set S. Moreover, L is a sublattice of C^S generated by a Boolean sublattice of C^S and the set D of diagonal elements of C^S. Conversely, any sublattice L of C^S which is generated by D and a Boolean sublattice B of C^S is the coproduct of D and B.*

Proof. Let S be the set of prime filters in B. Define a homomorphism $h_1\colon B \to C^S$ as follows. For each $a \in B$,

$$h_1(a)(F) = \begin{Bmatrix} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F \end{Bmatrix} \quad \text{for each } F \in S.$$

Define $h_2\colon C \to C^S$ by $h_2(c)(F) = c$ for each $F \in S$ and $c \in C$. Let $h\colon B * C \to C^S$ be the unique extension of h_1 and h_2 to $B * C$. Now suppose $h(x) \leqslant h(y)$, where $x, y \in B * C$. Then

$$x = \sum_{i=1}^{n} a_i e_i \quad \text{and} \quad y = \prod_{j=1}^{m} (b_j + c_j),$$

where $a_i, b_j \in B$; $e_i, c_j \in C$. We have for each i, j that $h_1(a_i)h_2(e_i) \leqslant h_1(b_j) + h_2(c_j)$. Suppose $a_i \nleqslant b_j$. Then there is a prime filter F in B such that $a_i \in F$, $b_j \notin F$. So

$$e_i = \big(h_1(a_i)h_2(e_i)\big)(F) \leqslant \big(h_1(b_j) + h_2(c_j)\big)(F) = c_j.$$

Thus $a_i e_i \leqslant b_j + c_j$ for each i, j and so $x \leqslant y$. Hence h is one to one. Clearly, $D = h(C)$, $h(B)$ is a Boolean sublattice of B and $D \cup h(B)$ generates $h(B*C)$. In particular, $h(B*C)$ is a subdirect product of B. Conversely, if B is a Boolean sublattice of C^S, then each $x \in C^S$ is such that $x(F)$ is 0 or 1 for each $F \in S$. Thus, suppose $a \in B$; $y, z \in D$ and $ay \leqslant z$. If $y \nleqslant z$, then, since y, z are diagonal elements, $y(F) \nleqslant z(F)$ for each $F \in S$. Thus, $a(F)y(F) \leqslant z(F)$ implies $a(F) \neq 1$, so $a(F) = 0$ for all F and hence $a = 0$.

Theorem 3.2 yields a structure theorem for Post algebras. Namely

A lattice L is a Post algebra if and only if, for some finite chain C and set S, L is a sublattice of C^S which is generated by D and a Boolean sublattice of C^S.

The next corollary of Theorem 3.2 is a generalization of Lemma 2.1.

COROLLARY 3.3. *If B is a Boolean algebra and C a chain, then B is the center of $B*C$.*

Proof. Suppose a is an element of the center of $B*C$, hence $h(a)$ is in the center of C^S and thus $h(a)(F) = 0$ or 1 for all $F \in S$. Let $a = \sum_{i=1}^{n} a_i e_i$ be the representation of a in the sense of (4). We may assume $a \neq 0$, hence $n \geqslant 2$. Suppose $n > 2$. Since $a_2 > a_3 > \ldots > a_n > 0$, there exists an $F_1 \in S$ such that $a_2 \in F_1$, $a_i \notin F_1$ for $3 \leqslant i \leqslant n$. Hence

$$h(a)(F_1) = \sum_{i=2}^{n} h_1(a_i)(F_1)e_i = e_2 \neq 0,$$

implying $e_2 = 1$ which contradicts $n > 2$. Thus $n = 2$ and $a = a_2 e_2$, $a_2 > 0$, $e_2 > 0$. It follows from (1′) that $a = a_2 \in B$.

For coproducts of a complete field of sets and a finite chain or equivalently, for Post algebras whose center is a complete field of sets, we have the following structure theorem.

THEOREM 3.4. *Let $B = 2^A$ be a complete field of sets and let C be a finite chain. Then $B*C$ is isomorphic to C^A. Conversely, if C is a finite chain and A a set, then C^A is the coproduct of its center and C. Hence a Post algebra is isomorphic to C^A, where C is its chain of constants and A is a set, if and only if the center of P is a complete field of sets.*

Proof. First suppose $B = 2^A$ and C is a finite set. We consider B to be the set of all subsets of the set A and alter the proof of Theorem 3.2 as follows. Define h_1: $B \to C^A$ by

$$h_1(x)(a) = \begin{cases} 1 & \text{if } \{a\} \subseteq x \\ 0 & \text{if } \{a\} \nsubseteq x \end{cases} \quad \text{for } x \in B,\ a \in A;$$

h_1 is a homomorphism, for $F_{\{a\}}$ is a prime filter in B. h_2: $C \to C^A$ is defined by $h_2(c)(a) = c$ for each $c \in C$, $a \in A$. Let h: $B*C \to C^A$ be the extension of h_1, h_2 to $B*C$. The proof that h is one to one is essentially the same as in Theorem 3.2. Finally let $f \epsilon C^A$. For each $c \epsilon C$, set $f^c = \{a \epsilon A | f(a) = c\}$. Then

$$h_1(f^c)(a) = \begin{cases} 1 & \text{if } f(a) = c \\ 0 & \text{otherwise} \end{cases};$$

hence

$$h\Big(\sum_{c \epsilon C} f^c \cdot c\Big)(a) = \sum_{c \epsilon C} \big(h_1(f^c) h_2(c)\big)(a) = f(a).$$

Thus $B*C \approx C^A$.

For the second part of the theorem, observe that the diagonals of C^A form a finite chain and that the center consists of all $f \epsilon C^A$ with $f(a) = 1$ or 0 for all $a \epsilon A$. It follows from (1) that C^A is the coproduct of its center and C.

The following corollary is an immediate consequence of Theorem 3.4.

COROLLARY 3.5. *Let B be a finite Boolean algebra, $B = 2^m$. Then for every finite chain C, $B*C \approx C^m$. If P is a Post algebra of order n whose center B is finite, $B = 2^m$, then $|P| = n^m$* (see also [9], p. 186).

4. Uniqueness of chains. One of the significant features of a Post algebra is the uniqueness of the chain of constants. Stated in terms of coproducts it says that the class $\mathscr{E}_1$ of finite chains C with the property that $B*C = B*C_1$ implies $C = C_1$ for all Boolean algebras B and finite chains C_1, contains all finite chains. This formulation raises an interesting question concerning uniqueness for coproducts of Boolean algebras and arbitrary chains. Precisely, the problem is to determine the class $\mathscr{E}$ of all chains C with the property that $B*C = B*C'$ implies $C = C'$ for every Boolean algebra B and every chain C'. We will prove in this section that $\mathscr{E}$ certainly does not contain all chains (Theorem 4.2) but on the other hand, $\mathscr{E}$ contains all finite chains (Theorem 4.1). This last result is obviously an improvement of the uniqueness property mentioned above. A further investigation of the class will be the subject of a subsequent paper.

THEOREM 4.1. *All finite chains belong to $\mathscr{E}$.*

Proof. Suppose C is a finite chain, C' an arbitrary chain and B a Boolean algebra such that $B*C = B*C'$. We must show $C = C'$. Clearly every element of C can be expressed in the form $\sum_{i=1}^{n} a_i c_i'$, where $a_i \epsilon B$, $c_i' \epsilon C'$ for $i = 1, \ldots, n$. But C is finite and hence there is a finite subchain C^+ of C' such that $B \cup C^+$ generates $B*C$. Then obviously $B*C = B*C^+$. But C and C^+ are finite, so $B*C$ and $B*C^+$ are Post algebras; hence $C = C^+ \subseteq C'$. Now let $c \epsilon C' \sim C$. Then $C^{++} = C \cup \{c\}$ is a chain that generates $B*C$. Thus $B*C = B*C^{++}$. This again implies, by the uniqueness property for Post algebras, that $C = C^{++}$. This contradiction establishes the result.

THEOREM 4.2. *Let C be a chain with a subchain S that satisfies*

(i) $\emptyset \neq S \subseteq C \sim \{0, 1\}$.

(ii) $x, y \in S$, $x \leqslant z \leqslant y$ *implies* $z \in S$.

(iii) *There is an isomorphism* $f\colon S \to S$ *such that* $x < f(x)$ *for all* $x \in S$ *or* $f(x) < x$ *for all* $x \in S$.

Then $C \notin \mathscr{E}$.

Proof. Suppose $x < f(x)$ for all $x \in S$. Let $B \neq 2$ be a Boolean algebra and $a \in B \sim \{0, 1\}$. Set $d_x = x + af(x)$ for each $x \in S$ and $C_1 = (C \sim S) \cup \cup \{d_x | x \in S\}$. We will show that C_1 is a subchain of the coproduct $B * C$ such that $B * C = B * C_1$ but $C \neq C_1$. Our first step is:

C_1 is a chain (with $0, 1$): Clearly $\{0, 1\} \subseteq C_1$ and u, v in C_1 are comparable if they are in $C \sim S$. Also if $x, y \in S$ and $x \leqslant y$, then $d_x \leqslant d_y$. Finally suppose $u \in C \sim S$ and $v = d_x$ for some $x \in S$. If $u < x$, then $u < x + af(x) = d_x$, and if $x < u$, then $u \nleqslant f(x)$ by (ii), so $f(x) \leqslant u$ and thus $d_x = x + af(x) \leqslant u$.

$B \cup C_1$ generates $B * C$: It is sufficient to show S is a subset of the lattice generated by $B \cup C_1$. So let $x \in S$, then by (iii), $x = x + f^{-1}(x) = \bar{a}x + f^{-1}(x) + ax = \bar{a}\big(x + af(x)\big) + f^{-1}(x) + ax = \bar{a}d_x + d_{f^{-1}(x)}$ which is in the sublattice generated by $B \cup C_1$.

$B * C = B * C_1$ and $C \neq C_1$: For the first part, it suffices to prove that if $p, q \in C_1$, $b \in B$ and $bp \leqslant q$, then $b = 0$ or $p \leqslant q$. Suppose $b \neq 0$. The result is immediate if $p, q \in C \sim S$. There are three remaining cases. First let $p \in C \sim S$, $q = d_x$ for some $x \in S$. Then

$$bp \leqslant x + af(x) \leqslant f(x), \text{ so } p \leqslant f(x).$$

By (ii) $x \nleqslant p$, so $p < x \leqslant d_x = q$. Next suppose $p = d_x$ for some $x \in S$ and $q \in C \sim S$. Then

$$bx \leqslant bx + ba\,f(x) = bd_x \leqslant q$$

implies $x \leqslant q$, so $q \nleqslant f(x)$.

Thus, $f(x) \leqslant q$ and $x + af(x) \leqslant q$. Finally, assume $p = d_x$ and $q = d_y$, where $x, y \in S$. Suppose $p \nleqslant q$. Then $x \nleqslant y$. Now

$$bx \leqslant bx + ab\,f(x) \leqslant bd_x \leqslant q \leqslant y + af(y) \leqslant y + a.$$

So $bx \leqslant a + y$ implies $b \leqslant a$ and $bf(x) \leqslant y + f(y) = f(y)$ implies $f(x) \leqslant f(y)$. But f is an isomorphism so $x \leqslant y$, a contradiction. Hence $B * C = B * C_1$. Now let $x \in S \subseteq C$. Then $d_x \in C_1$. To show $d_x \notin C$ suppose $c = x + af(x)$ for some $c \in C$. Then $x \leqslant c$ and $c \leqslant x + a$ and hence $c = x$. Also $af(x) \leqslant c$ implies $f(x) \leqslant c \leqslant x$, a contradiction. This completes the proof.

Example. Let G be any subgroup of the additive group R of the reals, $G \neq (0)$. Consider the chain $G^{\#}$ obtained by adjoining $-\infty$ and $+\infty$ to G. Let $S = G$. Then S satisfies the conditions of Theorem 4.2. Indeed, pick $a \in G$, $a \neq 0$, and let $f\colon S \to S$ be defined by $f(x) = x+a$, $x \in S$. It follows that $G^{\#} \notin \mathscr{E}$. In particular $R^{\#} \notin \mathscr{E}$ and therefore no closed interval $[a, b]$, $a \neq b$, of R belongs to $\mathscr{E}$.

REFERENCES

[1] B. Banaschewski and B. Bruns, *Categorical characterization of the MacNeille completion*, Archiv der Mathematik 18 (1967), p. 369-377.

[2] — *Injective hulls of distributive lattices*, Journal für die reine und angewandte Mathematik 232 (1968), p. 102-109.

[3] C. C. Chang and A. Horn, *Prime ideal characterization of generalized Post algebras*, *Lattice Theory*, ed. Dilworth (1960), p. 43-48.

[4] Ph. Dwinger, *Notes on Post algebras I, II*, Indagationes Mathematicae 28 (1966), p. 462-478.

[5] — *Generalized Post algebras*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 16 (1968), p. 559-565.

[6] G. Epstein, *The lattice theory of Post algebras*, Transactions of the American Mathematical Society 95 (1960), p. 300-317.

[7] G. Grätzer, *Universal algebra*, Van Nostrand (1968).

[8] G. Grätzer and H. Lakser, *Chain conditions in the distributive free product of lattices*, Transactions of the American Mathematical Society 144 (1969), p. 301-312.

[9] P. Rosenbloom, *Post algebras. I Postulates and general theory*, American Journal of Mathematics 64 (1942), p. 167-183.

[10] T. G. Rousseau, *Post algebras and pseudo-Post algebras*, Fundamenta Mathematicae 67 (1970), p. 133-145.

[11] T. Traczyk, *On axioms and some properties of Post algebras*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 10 (1963), p. 193-209.

[12] — *On Post algebras, with uncountable chain of constants. Algebras and homomorphisms*, ibidem 15 (1967), p. 673-680.

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