C O L L O Q U I U M M A T H E M A T I C U M

VOL. XLIV 1981 FASC. 1

BALAYAGE BY FOURIER TRANSFORMS WITH SPARSE FREQUENCIES IN COMPACT ABELIAN GROUPS

BY

GEORGE S. SHAPIRO (BROOKLYN, NEW YORK)

1. Introduction and general results. Throughout, Γ will denote an infinite compact abelian group with (discrete) dual group G, both written additively. If $E \subset \Gamma$ is compact and $\Lambda \subset G$, then $A_\Lambda(E)$ denotes the set of functions φ on E expressible in the form

$$\varphi(t) = \sum_{\lambda \in \Lambda} a_\lambda \langle \lambda, t\rangle \quad \text{for all } t \in E \tag{1}$$

with $\sum |a_\lambda| < \infty$. The set $A_\Lambda(E)$ is a Banach space under the norm

$$\|\varphi\|_{A_\Lambda(E)} = \inf\Big\{\sum |a_\lambda|: (1) \text{ holds}\Big\}.$$

We write $A(E)$ for $A_G(E)$ and recall a notion of Kahane [4], p. 150:

Definition 1. E is an AA_Λ if $A(E) = A_\Lambda(E)$.

In this situation (though for the case where Γ is not compact) Beurling [1] (see also [7]) says that *balayage is possible for* (Λ, E).

Definition 2. E is a *BAS set* (BAS stands for *balayage with arbitrarily sparse frequencies*) if, given any sequence of functions $\{F_n\}_1^\infty$ having values in the collection of finite subsets of G, there is a set $\Lambda = \{\lambda_n\}_1^\infty$ such that E is an AA_Λ and such that

$$\lambda_n \notin F_n(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \quad \text{for all } n. \tag{2}$$

(Here, F_1 is just a finite subset of G.)

Kahane [4], p. 151, and [5], p. 160, showed the existence of fairly large BAS sets (of Cantor type) in the circle group. In [7] similar examples were constructed in the real line by using a different technique. The author [8] used a related method to construct BAS sets in a class of compact groups (see below). In this paper, we show the existence of reasonably large BAS sets in all metrizable compact abelian groups.

If S is a subset of a group and m a positive integer, then mS denotes the set of sums $\{x_1+x_2+\ldots+x_m: x_i \in S,\ i=1,2,\ldots,m\}$. Our main result is

THEOREM 1. *If Γ is a metrizable compact abelian group, then Γ contains a BAS set E such that $6E = \Gamma$.*

Before proving Theorem 1, we need some auxiliary results. For the rest of the paper, Γ is assumed to be metrizable so that G is countable.

Definition 3. $\{x_n\}_1^\infty \subset G$ is a *unity-approximating* (*UA*) *sequence* for $E \subset \Gamma$ if the x_n are distinct and, for all n and all $t \in E$,

$$|1-\langle x_n, t\rangle| \leqslant 2\sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{2}.$$

The arguments in [7], p. 194 and 195, can be generalized and (since G is discrete and countable) simplified to prove

THEOREM 2. *If there is a UA sequence for E, then E is a BAS set.*

Proof. Let $\{x_n\}_1^\infty$ be a UA sequence for E. Enumerate $G = \{g_n\}_1^\infty$. Given a sequence of functions $\{F_n\}_1^\infty$ as in Definition 2, we define $\Lambda = \{\lambda_n\}_1^\infty$ iteratively as follows. Assuming $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ to have been chosen, there is some k_n such that

$$x_{k_n} \notin F_n(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) - g_n$$

since that set is finite. Let $\lambda_n = x_{k_n} + g_n$. By construction, Λ satisfies (2). We must show that E is an AA_Λ.

The argument given in [6], p. 107 and 108, shows that the condition

$$\sup\{|1-\langle x, t\rangle|: t \in E\} \leqslant 2\sin\frac{\pi}{2m}$$

for m an odd integer implies that

$$\|1-\langle x, \cdot\rangle\|_{A(E)} \leqslant 2\sin\frac{\pi}{2m}$$

and, thus,

$$\|1-\langle x_n, \cdot\rangle\|_{A(E)} \leqslant 2\sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{2} \quad \text{for all } n.$$

Since $\langle g_n, t\rangle - \langle \lambda_n, t\rangle = \langle g_n, t\rangle(1-\langle x_{k_n}, t\rangle)$, we have also

$$\|\langle g_n, \cdot\rangle - \langle \lambda_n, \cdot\rangle\|_{A(E)} \leqslant \frac{\sqrt{5}-1}{2}.$$

Now, given $\varphi \in A(E)$ and $\varepsilon > 0$, there are $\{a_n\}$ with

$$\sum |a_n| \leqslant (1+\varepsilon)\|\varphi\|_{A(E)} \quad \text{and} \quad \varphi(t) = \sum a_n\langle g_n, t\rangle \text{ for } t \in E.$$

Define ψ by $\psi(t) = \sum a_n\langle\lambda_n, t\rangle$ for $t \in E$. Clearly,

$$\psi \in A_\Lambda(E) \quad \text{and} \quad \|\psi\|_{A_\Lambda(E)} \leqslant (1+\varepsilon)\|\varphi\|_{A(E)}.$$

We have

$$\|\varphi - \psi\|_{A(E)} \leqslant \sum |a_n|\, \|\langle g_n, \cdot\rangle - \langle\lambda_n, \cdot\rangle\|_{A(E)} \leqslant (1+\varepsilon)\frac{\sqrt{5}-1}{2}\|\varphi\|_{A(E)}.$$

A standard iteration now shows that $\varphi \in A_\Lambda(E)$, so that E is an AA_Λ and the theorem follows.

We say that K is a *quotient* of Γ if there is a closed subgroup $\Gamma_0 \subset \Gamma$ such that K is topologically isomorphic to Γ/Γ_0. If H is a subgroup of G, we write $H^\perp$ for the annihilator of H,

$$H^\perp = \{t \in \Gamma \colon \langle x, t\rangle = 1 \text{ for all } x \in H\},$$

and we recall that $\Gamma/H^\perp \simeq \hat{H}$, the dual group of H. Let Z_p denote the group of p-adic integers, and T the circle group. In order to reduce the proof of Theorem 1 to constructions in certain specific groups, we need a result which is similar, but not identical, in both statement and proof to a result of Varopoulos [9], Lemma 6.2.

THEOREM 3. *Any compact, metrizable abelian group Γ satisfies at least one of the following*:

(a) *Γ is topologically isomorphic to a group of the form $\mathop{\times}\limits_{1}^{\infty} \Gamma_n$, where each Γ_n is compact and non-trivial.*

(b) *T is a quotient of Γ.*

(c) *Z_p is a quotient of Γ for some prime p.*

Proof. Consider the countable abelian group G. If G contains an element x of infinite order, then $H = \{nx \colon n \in Z\}$ is isomorphic to Z so that $\Gamma/H^\perp \simeq \hat{H} \simeq \hat{Z} = T$ and (b) holds.

Thus, we may assume that G is a countable torsion group. By [2], 21.3, we have $G = D \oplus B$, where D is a divisible subgroup of G and B is reduced, i.e., B contains no non-trivial divisible subgroups. If D is non-trivial, then, being a divisible torsion group, D (and hence G) contains a subgroup H isomorphic to $Z(p^\infty)$ for some prime p ([2], 23.1). (See Section 2 for a discussion of this and related groups.) Then $\Gamma/H^\perp \simeq Z(p^\infty)\hat{}$ $\simeq Z_p$ and (c) holds.

The situation remains where G is itself reduced. In this case, write

$$G = \bigoplus_p G_p,$$

where G_p is the p-primary component of G and the sum is over all primes ([2], 8.4). If infinitely many of the G_p are non-trivial, then dualizing shows that (a) holds. Otherwise, there is a p_0 such that G_{p_0} is a countably infinite reduced p_0-group and is, therefore, itself the sum of $\aleph_0$ non-trivial p_0-groups ([2], 77.5). This clearly implies that G is the sum of $\aleph_0$ non-trivial subgroups, so that dualizing again shows that (a) holds, proving the theorem.

We state two simple results which we shall need.

LEMMA 1. *Let K be a closed subgroup of Γ and let the projection π: $\Gamma \to \Gamma_0 \simeq \Gamma/K$ be canonical. Regard $K^{\perp} \subset G \; (= \hat{\Gamma})$ as the dual group of Γ_0. Let $E \subset \Gamma_0$ and suppose that $\{x_n\}_1^{\infty} \subset K^{\perp}$ is a UA sequence for E. Then $\{x_n\}_1^{\infty}$, regarded as a sequence in G, is also a UA sequence for $\pi^{-1}(E)$.*

Proof. The result follows immediately from the observation that if $x \in K^{\perp}$ and $\gamma \in \Gamma$, then

$$\langle x, \gamma \rangle = \langle x, \pi(\gamma) \rangle,$$

where, on the left, we consider the pairing between G and Γ and, on the right, that between $K^{\perp}$ and Γ_0.

LEMMA 2. *If Γ_0 is a quotient of Γ, π is the canonical projection and $E \subset \Gamma_0$ satisfies $mE = \Gamma_0$, then $m\pi^{-1}(E) = \Gamma$.*

Proof. Given $\gamma \in \Gamma$, there are, by hypothesis, $t_1, t_2, \ldots, t_m \in E$ with $\pi(\gamma) = t_1 + t_2 + \ldots + t_m$. Since π is surjective, there are $\gamma_1, \gamma_2, \ldots, \gamma_m \in \Gamma$ with $\pi(\gamma_i) = t_i$ for $i = 1, 2, \ldots, m$. Thus, if

$$\sigma = \gamma - (\gamma_1 + \gamma_2 + \ldots + \gamma_m),$$

then $\pi(\sigma) = 0$ and $\pi(\gamma_1 + \sigma) = t_1$. The equation $\gamma = (\gamma_1 + \sigma) + \gamma_2 + \ldots + \gamma_m$, therefore, exhibits γ as an element of $m\pi^{-1}(E)$.

In [8] it was shown that if Γ is of the type described in (a) of Theorem 3, then there is a BAS set E (containing 0) in Γ with $3E = \Gamma$. In view of Theorem 3 and Lemmas 1 and 2, it suffices for the proof of Theorem 1 to consider only the group T and the various groups Z_p. Accordingly, we shall construct, for any p, a compact set $E \subset Z_p$ containing a UA sequence and such that $6E = Z_p$. The outline of such a construction for T will then be indicated. (In [8] a similar construction for R is less clearly presented.)

2. Construction in Z_p and in T. We first introduce some notation and recall certain facts. $I(m)$ denotes the group of integers modulo m regarded as the set $\{0, 1, 2, \ldots, m-1\}$. Let p be a fixed prime and $q = p^r$, where r is a positive integer to be specified later. For any r, the group Z_p of p-adic integers is isomorphic to the group Z_q described below but, for technical reasons, it will be more convenient to work with Z_q.

An element $t \in Z_q$ is regarded as a sequence

$$t = \langle a_0, a_1, a_2, \ldots \rangle, \quad \text{where } a_i \in I(q) \text{ for all } i. \tag{3}$$

Z_q is given the product topology and we think of t in (3) as the convergent series $\sum_0^\infty a_k q^k$, so that the addition of elements is not componentwise addition in the product $I(m)^{\aleph_0}$, rather it "carries to the right". (For a detailed treatment of Z_q in somewhat different notation see [3], Sections 10.1-10.10, 25.1, 25.2.)

For any $k = 1, 2, 3, \ldots$ with t as in (3) we define

$$\sigma_k(t) = a_0 + a_1 q + \ldots + a_{k-1} q^{k-1}$$

so that σ_k is a homomorphism of Z_q onto $I(q^k)$.

$Z(q^\infty)$ ($\simeq Z(p^\infty)$) is the group of rational numbers of the form m/q^k with addition performed modulo one. $Z(q^\infty)$ may be regarded as the dual group of Z_q under the pairing

$$\left\langle \frac{m}{q^k}, t \right\rangle = \exp\left(2\pi i \frac{m}{q^k} \sigma_k(t)\right).$$

We now assume that r is picked so that $q = p^r \geqslant 81$. Then it is always possible to find an integer s such that

$$\frac{q}{10} \geqslant s \tag{4}$$

and

$$s \geqslant \frac{q+6}{12}. \tag{5}$$

Having chosen such an s, we let E be the set of

$$t = \langle a_0, a_1, a_2, \ldots \rangle \in Z_q$$

such that

$$a_i \in B_s = \{0, 1, \ldots, s-1\} \cup \{q-s, q-s+1, \ldots, q-1\}$$

for all i. One verifies readily that (as subsets of the group $I(q)$)

$$mB_s = \{0, 1, \ldots, m(s-1)\} \cup \{q-ms, q-ms+1, \ldots, q-1\},$$

so that the inequality $6(s-1) \geqslant q - 6s$ (which follows from (5)) implies that $6B_s = I(q)$. Using this fact and given $t \in Z_q$, one can construct iteratively $t_1, t_2, \ldots, t_6 \in E$, a digit at a time (taking account of the "carrying") such that $t = t_1 + t_2 + \ldots + t_6$. Thus $6E = Z_q$.

Suppose that $t = \langle a_0, a_1, a_2, \ldots \rangle \in E$. If $a_{k-1} \leqslant s-1$, then $\sigma_k(t) \leqslant sq^{k-1} - 1$, whereas if $a_{k-1} \geqslant q - s$, then $\sigma_k(t) \geqslant q^k - sq^{k-1}$. In either case, $\sigma_k(t)/q^k$ is within s/q of an integer (either 0 or 1) so that

$$\left|1 - \left\langle \frac{1}{q^k}, t \right\rangle\right| = \left|1 - \exp\left(2\pi i \frac{\sigma_k(t)}{q^k}\right)\right|$$

$$\leqslant \left|1 - \exp\left(2\pi i \frac{s}{q}\right)\right| \leqslant \left|1 - \exp\left(\frac{2\pi i}{10}\right)\right| = 2 \sin \frac{\pi}{10}$$

because (by (4)) $s/q \leqslant 1/10$. Thus, we have shown that $\{1/q^k\}_1^\infty$ is a UA sequence for E so that the proof of Theorem 1 in case (c) of Theorem 3 is complete.

We turn now to the circle group T which we realize, for convenience, as the set of real numbers t such that $0 \leqslant t < 1$ with addition performed modulo one. In this realization, the pairing of Z and T is given by $\langle n, t\rangle = \exp(2\pi int)$. Pick integers q and s satisfying (4) and (5) (q need not be a prime power). Then we let E be the set of $t \in T$ expressible to base q in the form

$$t = \sum_1^\infty a_k q^{-k},$$

where $a_k \in \{0, 1, \ldots, s-1\} \cup \{q-s, q-s+1, \ldots, q-1\}$ for all k.

It is readily established (as above, by (4)) that $\{q^k\}_1^\infty \subset Z$ is a UA sequence for E. The equality $6E = T$ follows, again, from (5). (The details differ since now addition carries to the left. Given

$$t = \sum_1^\infty a_k q^{-k} \in T,$$

one shows by induction on n that, for each n,

$$\sum_1^n a_k q^{-k} \in 6E,$$

and then uses the fact that $6E$ is closed.) This completes the proof of Theorem 1.

In the light of Theorem 1 and the result of [8], the following question arises:

Are there any BAS sets E in any Γ (and, especially, in $\Gamma = T$) with $E+E = \Gamma$? **(P 1198)**

REFERENCES

[1] A. Beurling, *On balayage of measures in Fourier transforms*, Notes from a seminar at the Institute for Advanced Study, Princeton, New Jersey, 1959-1969 (unpublished).

[2] L. Fuchs, *Infinite abelian groups*, Vol. I and II, Pure and Applied Mathematics 36 (1970 and 1973).

[3] E. Hewitt and K. A. Ross, *Abstract harmonic analysis, I*, Die Grundlehren der mathematischen Wissenschaften 115 (1963).

[4] J.-P. Kahane, *Séries de Fourier absolument convergentes*, Ergebnisse der Mathematik und ihrer Grenzgebiete 50 (1970).

[5] — and R. Salem, *Ensembles parfaits et séries trigonométriques*, Actualités des Sciences Industrielles 1301 (1963).

[6] Y. Meyer, *Algebraic numbers and harmonic analysis*, North-Holland Mathematical Library, Vol. II, Amsterdam 1972.
[7] G. Shapiro, *Balayage in Fourier transforms: general results, perturbation and balayage with sparse frequencies*, Transactions of the American Mathematical Society 225 (1977), p. 183-198.
[8] — *Balayage by Fourier transforms with sparse frequencies in compact abelian torsion groups*, Proceedings of the American Mathematical Society 71 (1978), p. 253-256.
[9] N. Th. Varopoulos, *Sets of multiplicity in locally compact abelian groups*, Annales de l'Institut Fourier, Grenoble, 16 (1966), p. 123-158.

DEPARTMENT OF MATHEMATICS
BROOKLYN COLLEGE
BROOKLYN, NEW YORK

Reçu par la Rédaction le 20. 7. 1978