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IRREDUCIBLY CONFLUENT MAPPINGS

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All spaces in this paper are assumed to be compact and metric. A *continuum* is a compact, connected metric space. A *mapping* $f\colon X \to Y$ is a continuous function from X to Y. A mapping $f\colon X \to Y$ is said to be *confluent* ([2], p. 213) if, for each subcontinuum K of Y and each component C of $f^{-1}(K)$, $f(C) = K$. Whyburn [8] showed that monotone mappings and open mappings are confluent. If $f\colon X \to Y$ is a mapping of X onto Y such that if $y \in Y$, C is a component of $f^{-1}(y)$, and U is an open set containing C, then $y \in \operatorname{Int}(f(U))$, then f is said to be *quasi-interior* [6]. Clearly, every open mapping is quasi-interior. Lelek and Read [ibidem] showed that every quasi-interior mapping is confluent. A confluent (respectively, quasi-interior) mapping $f\colon X \to Y$ from X onto Y is said to be *irreducibly confluent* (respectively, *irreducibly quasi-interior*) if there does not exist a proper subcontinuum H of X such that $f|H$ is a confluent (respectively, quasi-interior) mapping of H onto Y.

It is well known [7] that if $f\colon X \to Y$ is a confluent mapping of a space X onto a continuum Y, then there is a subcontinuum L of X such that $f|L$ is an irreducibly confluent mapping of L onto Y. In this paper it is shown that an analogous statement cannot be made for quasi-interior mappings. Also, conditions under which certain irreducibly confluent mappings are monotone or irreducible are developed.

A mapping $f\colon X \to Y$ is said to be *locally confluent* [4] if for each point y in Y there is an open subset O of Y containing y such that $f|f^{-1}(\bar{O})$ is confluent. It has been shown [6] that all locally confluent mappings onto locally connected spaces are quasi-interior.

THEOREM 1. *If $f\colon X \to Y$ is a locally confluent mapping of X onto a locally connected continuum Y, then there is a subcontinuum L of X such that $f|L$ is an irreducibly quasi-interior mapping of L onto Y.*

Proof. Since f is locally confluent, it follows that f is quasi-interior and thus confluent. Hence, there is a subcontinuum L of X such that $f|L$ is an irreducibly confluent mapping of L onto Y. Since $f|L$ is locally confluent, it follows that $f|L$ is quasi-interior. If K is a subcontinuum of L such that $f|K$ is a quasi-interior mapping of K onto Y, then $f|K$ is confluent, so $K = L$. Hence $f|L$ is irreducibly quasi-interior.

COROLLARY. *If $f: X \to Y$ is a quasi-interior mapping of X onto a locally connected continuum Y, then there is a subcontinuum L of X such that $f|L$ is an irreducibly quasi-interior mapping of L onto Y.*

The following example shows that, in general, such a subcontinuum L need not exist:

Example 1. Let $p = (0, 1)$, $q_0 = (-1, 0)$, and $r_0 = (1, 0)$ in the Euclidean plane E^2. For each positive integer n, let $q_n = (-1-1/n, 0)$ and $r_n = (1+1/n, 0)$. For $x, y \in E^2$, denote the line segment joining x and y by $[x, y]$. For $x = (a, b) \in E^2$, denote the reflection of x in the vertical axis (i.e. the point $(-a, b)$) by $\mathrm{Ref}(x)$. For each non-negative integer n, let

$$I_n = [p, q_n] \quad \text{and} \quad J_n = [p, r_n].$$

Let

$$A = [q_0, r_0]$$

and

$$B = \{(a, b):\ -1 \leqslant a \leqslant 1,\ b \leqslant 0,\ \text{and}\ a^2+b^2 = 1\}.$$

Let

$$I = \bigcup_{n=0}^{\infty} I_n, \quad J = \bigcup_{n=0}^{\infty} J_n \quad \text{and} \quad X = A \cup B \cup I \cup J.$$

Clearly, X is a continuum. Define an equivalence relation R on X by

$$\begin{aligned} R = \{(x, x):\ x \in X\} &\cup \{(x, y):\ x \in I_n,\ y \in J_n \text{ for some non-negative integer } n, \text{ and } y = \mathrm{Ref}(x)\} \cup \\ &\cup \{(y, x):\ x \in I_n,\ y \in J_n \text{ for some non-negative integer } n, \text{ and } y = \mathrm{Ref}(x)\} \cup \\ &\cup \{(x, y):\ x \in A \text{ and } y \in B \cap L(x)\} \cup \{(y, x):\ x \in A \text{ and } y \in B \cap L(x)\}, \end{aligned}$$

where $L(x)$ denotes the line through p and $\mathrm{Ref}(x)$.

It is easy to see that R determines a lower semi-continuous decomposition of X, and thus the natural projection $f: X \to Y$ of X onto the quotient space $Y = X/R$ is an open mapping. If L is a subcontinuum of X such that $f|L$ is a quasi-interior mapping of L onto Y, then there must exist a positive integer n such that if $m > n$, then $I_m \cup J_m \subset L$. But $L_1 = (L \setminus I_{n+1}) \cup \{p\}$ is a proper subcontinuum of L such that $f|L_1$ is a quasi-interior mapping of L_1 onto Y. Hence there does not exist a subcontinuum L of X such that $f|L$ is an irreducibly quasi-interior mapping of L onto Y.

QUESTION. Can such an example be constructed with X hereditarily unicoherent? (**P 956**) (A continuum is said to be *hereditarily unicoherent* if the intersection of each pair of its subcontinua is either a continuum or empty.)

A *dendrite* is a hereditarily unicoherent, hereditarily locally connected continuum (cf. [8], p. 88). The following theorem shows that there is a class of continua having the property that any irreducibly confluent mapping from a continuum onto one of these continua must be monotone:

THEOREM 2. *If $f\colon X \to Y$ is an irreducibly confluent mapping of a continuum X onto a dendrite Y, then f is monotone.*

Proof. Since Y is locally connected, f is quasi-interior. It has been shown (see Corollary 3.1 of Lelek and Read [6]) that f is quasi-interior if and only if f factors in the form $f = hg$, where g is monotone and h is light and open. Thus, since Y is a dendrite, and $h\colon g(X) \to Y$ is light and open, there exists a dendrite D contained in $g(X)$ such that $h|D$ is a homeomorphism (see [8], p. 188). Hence

$$f|g^{-1}(D)\colon\ g^{-1}(D) \to Y$$

is the composition of a monotone mapping and a homeomorphism, and is, therefore, monotone. Clearly, since f is irreducibly confluent, $g^{-1}(D) = X$, so f is monotone.

The following two examples show that neither hereditary unicoherence nor hereditary local connectedness can be left out of the hypothesis of Theorem 2.

Example 2. Let X and f be as in Example 1. Let H be the subcontinuum of X defined by $H = A \cup B \cup I_0 \cup J_0$. Then $g = f|H$ is an open mapping of H onto $f(H)$ such that g maps no proper subcontinuum of H confluently onto $f(H)$. Thus, g is an irreducibly confluent mapping of H onto $f(H)$ which is not monotone, even though $f(H)$ is hereditarily locally connected.

Example 3. In E^2 let

$$X = \{(a,1)\colon\ -1 \leqslant a \leqslant 1\} \cup \{(1,b)\colon\ -1 \leqslant b \leqslant 1\} \cup \cup \left\{\left(a, \sin\frac{1}{a-1}\right)\colon\ 1 < a \leqslant 2\right\} \cup \left\{\left(\sin\frac{1}{b-1}, b\right)\colon\ 1 < b \leqslant 2\right\}.$$

Let R be the equivalence relation such that

$$R = \{\big((t,1),(1,t)\big)\colon\ -1 \leqslant t \leqslant 1\} \cup \{\big((1,t),(t,1)\big)\colon\ -1 \leqslant t \leqslant 1\} \cup \cup \{(z,z)\colon\ z \in X\}.$$

Clearly, X and $Y = X/R$ are arc-like continua, so Y is hereditarily unicoherent [1]. Further, it is easily seen that the natural projection mapping $f\colon X \to Y$ is an irreducibly confluent mapping which is not monotone.

It is known ([5], p. 171) that if $f: X \to Y$ is a mapping of a continuum X onto a continuum Y, then there is a subcontinuum K of X such that $f|K$ is *irreducible* in the sense that $f(K) = Y$, but f maps no proper subcontinuum of K onto Y. Certain classes of continua have the property that any irreducibly confluent mapping onto one of these continua is always irreducible. A continuum is *hereditarily indecomposable* if, for each pair H, K of its non-degenerate subcontinua, $H \cap K \neq \emptyset$ implies that H is contained in K or K is contained in H.

THEOREM 3. *If $f: X \to Y$ is an irreducibly confluent mapping of a continuum X onto a hereditarily indecomposable continuum Y, then f is irreducible.*

Proof. Suppose, by way of contradiction, that there is a proper subcontinuum L of X such that $f(L) = Y$. By a result of Cook ([3], p. 243), $f|L$ is confluent, which is the desired contradiction.

LEMMA. *If $f: X \to Y$ is a confluent mapping of the hereditarily indecomposable continuum X onto Y, then Y is hereditarily indecomposable.*

Proof. Let H and K be non-degenerate subcontinua of Y such that $H \cap K \neq \emptyset$. Let $p \in H \cap K$ and $x \in f^{-1}(p)$. Let A be the component of $f^{-1}(H)$ containing x, and let B be the component of $f^{-1}(K)$ containing x. Then $f(A) = H$ and $f(B) = K$. Further, since $x \in A \cap B$, either $A \subset B$ or $B \subset A$. Hence, $H \subset K$ or $K \subset H$. Thus Y is hereditarily indecomposable.

The Lemma together with Theorem 3 immediately imply the following theorem:

THEOREM 4. *An irreducibly confluent mapping $f: X \to Y$ from a hereditarily indecomposable continuum X onto a continuum Y is irreducible.*

THEOREM 5. *If $f: X \to Y$ is an irreducibly confluent mapping of a hereditarily unicoherent continuum X onto a dendrite Y, then f is irreducible.*

Proof. Let f be an irreducibly confluent mapping from X onto Y. By Theorem 2, f is monotone. Suppose, by way of contradiction, that there is a proper subcontinuum K of X such that $f(K) = Y$. Then, for each $y \in Y$, $f^{-1}(y)$ is a continuum. Thus, for each $y \in Y$, $K \cap f^{-1}(y)$ is a subcontinuum of K. Hence $f|K$ is monotone and thus confluent, which is the desired contradiction. Therefore, f is irreducible.

A somewhat stronger result can be obtained if X and Y are both arcs.

THEOREM 6. *If $f: X \to Y$ is a confluent mapping of an arc X onto an arc Y, and H is a subcontinuum of X such that $f|H$ is irreducible, then $f|H$ is irreducibly confluent.*

Proof. Suppose, without loss of generality, that $X = Y = [0, 1]$. Let $H = [a, b]$ be a proper subcontinuum of X such that $f|H$ is irreducible. Suppose, by way of contradiction, that $a \notin f^{-1}(\{0, 1\})$. Then there is a $c \in (a, b]$ such that $c \in f^{-1}(\{0, 1\})$. Hence, either $f([a, c]) = Y$ or $f([c, b]) = Y$, which is a contradiction. Thus $a \in f^{-1}(\{0, 1\})$. A similar argument shows that $b \in f^{-1}(\{0, 1\})$. Clearly, $f(a) \neq f(b)$. (Otherwise choose a point $z \in (a, b)$ such that $f(z) = 1$ if $f(a) = 0$, or such that $f(z) = 0$ if $f(a) = 1$. Then $f([a, z]) = Y$.) No generality is lost by assuming that $f(a) = 0$ and $f(b) = 1$.

Let $g = f|H$. Suppose, by way of contradiction, that g is not monotone. Then there is a $p \in Y$ such that $g^{-1}(p)$ is not connected. Let A and B be different components of $g^{-1}(p)$, say $A = [r, s]$ and $B = [t, u]$ with $s < t$. Clearly,

$$[s, t] \cap f^{-1}(\{0, 1\}) = \emptyset,$$

for if $z \in [s, t]$ and $f(z) = 0$, then $f([z, b]) = Y$ which is a contradiction. An analogous argument shows that

$$[s, t] \cap f^{-1}(1) = \emptyset.$$

Further, $f([s, t])$ is a non-degenerate subcontinuum of Y, say $f([s, t]) = [v, w]$ with $p < w < 1$ or $0 < v < p$. For $p < w$, let x be an element of $[s, t] \cap f^{-1}(w)$ and let C be the component of $f^{-1}\big([w, (w+1)/2]\big)$ containing x. Since f is confluent,

$$f(C) = \left[w, \frac{w+1}{2}\right].$$

But this is a contradiction, since $C \cap f^{-1}(p) = \emptyset$ implies that $C \subset (s, t)$, so $f(C) \subset f([s, t]) = [v, w]$. A similar contradiction is reached if $v < p$. Hence g is monotone, and, therefore, confluent. Thus $f|H$ is irreducibly confluent.

Theorems 3 and 5 show that irreducibly confluent mappings onto certain types of continua are irreducible. The following example shows that this is not generally the case, even for relatively "nice" continua:

Example 4. In E^2 let $p = (0, 1)$, $q = (1, 0)$, $a = (1, -1)$, and $b = (1, 1)$. Let

$$X = [p, q] \cup [p, b] \cup [a, b] \cup \left\{\left(x, \sin\frac{1}{x-1}\right):\ 1 < x \leqslant 2\right\} \cup$$

$$\cup \left\{\left(\frac{1}{2} + \frac{1}{2}\sin\frac{1}{y-1}, y\right):\ 1 < y \leqslant 2\right\}.$$

Define an equivalence relation R on X by

$$R = \big\{\big((1, t), (t, 1)\big):\ t \geqslant 0\big\} \cup \big\{\big((t, 1), (1, t)\big):\ t \geqslant 0\big\} \cup$$

$$\cup \{(r, s):\ r \in [p, q] \text{ and } s \in [p, q]\} \cup \{(r, r):\ r \in X\}.$$

Let Y be the quotient space X/R and let f be the natural projection mapping from X onto $Y = X/R$. It is easily seen that Y is an arc-like continuum and f is irreducibly confluent. It is not the case, however, where f is irreducible, since

$$f(\{p, q\} \cup (X \setminus [p, q])) = Y.$$

The idea used in Example 4 can be modified to produce an example of an irreducibly confluent mapping onto a dendroid (i.e., an arcwise connected, hereditarily unicoherent continuum) which is not irreducible. In each of these examples the domain of the mapping fails to be unicoherent.

QUESTION. If f is an irreducibly confluent mapping from a hereditarily unicoherent continuum onto an arc-like continuum, then is f irreducible? (**P 957**)

The following example shows that Theorem 6 cannot be generalized to arbitrary dendrites.

Example 5. In E^2 let $a = (0, 0)$, $b = (1, 0)$, $b' = (-1, 0)$, $c = (1, 1)$, $c' = (-1, 1)$, $d = (2, 0)$, and $d' = (-2, 0)$. Write

$$X = [d', d] \cup [b', c'] \cup [b, c] \quad \text{and} \quad Y = [a, b] \cup [b, c] \cup [b, d],$$

and define $f\colon X \to Y$ by $f((x, y)) = (|x|, y)$. Then f is confluent. Further, $f|[d', b] \cup [b, c]$ is an irreducible mapping of $[d', b] \cup [b, c]$ onto Y which is not confluent.

If $f\colon X \to Y$ is a mapping from a continuum X onto a hereditarily indecomposable continuum Y, then X contains an indecomposable continuum ([5], p. 208). Thus, X has to contain uncountably many indecomposable continua, e.g., each component of the preimage of any non-degenerate subcontinuum of Y must contain an indecomposable continuum. Thus, it seems reasonable to expect that the preimage, under an irreducibly confluent mapping, of a hereditarily indecomposable continuum might be hereditarily indecomposable. The following example, however, shows that this is not the case:

Example 6. Let X be a hereditarily indecomposable continuum, K a proper subcontinuum of X, and p and q members of K such that K is irreducible between p and q. Define an equivalence relation R on X by

$$R = \{(p, q), (q, p)\} \cup \{(x, x)\colon x \in X\}.$$

Let Z be the quotient space X/R and let $g\colon X \to Z = X/R$ be the natural projection mapping. Now define an equivalence relation S on Z by

$$S = \{(z, w)\colon z \in g(K) \text{ and } w \in g(K)\} \cup \{(z, z)\colon z \in Z\}.$$

Let $Y = Z/S$ and let f be the (monotone) natural projection mapping. Then fg is monotone, so, by the Lemma preceding Theorem 4, Y is hereditarily indecomposable. Thus f is confluent ([3], p. 243). Further, fg is one-to-one on $X \setminus K$, so $X \setminus K$ must be contained in any subset of X which is mapped onto Y. Hence, since K is nowhere dense ([5], p. 207), the only closed subset of X which fg maps onto Y is X itself. Thus if H is a subcontinuum of Z such that $f(H) = Y$, then $g^{-1}(H) = X$, so $H = Z$. Hence f is irreducibly confluent. Now let U and V be open subsets of X such that $\bar{U} \cap \bar{V} = \emptyset$, with $p \in U$ and $q \in V$. Let L be the closure of the component of U which contains p, and let M be the closure of the component of V which contains q. Then $g(L \cup M) = g(L) \cup g(M)$ is a non-degenerate subcontinuum of Z with $g(L) \cap g(M) = \{g(p)\}$. Hence Z is not hereditarily indecomposable.

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