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A REMARK ON EXPANDING MAPPINGS

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It is known that for any expanding mappings of class C^r $(r \geqslant 2)$ of a compact, connected differentiable manifold there exists an invariant normalized measure of class C^{r-1} [2]. In this note we shall prove that the assumption on r is essential.

In the sequel the following notation and terminology will be used:

M — a compact, connected differentiable manifold of class C^∞;

$\|\cdot\|$ — a Riemannian metric of class C^∞ on M;

ϱ — the natural distance induced by $\|\cdot\|$;

$\mathfrak{B}$ — the family of all Borel subsets of M;

$\mathfrak{M}^0$ — the set of all normalized measures of class C^0 on M, i.e., of all normalized Borel measures ν on M such that for each chart (U, α) in M there exists a positive continuous function $g_{\nu,\alpha}$ on $\alpha(U)$ such that

$$\nu(A) = \int_{\alpha(A)} g_{\nu,\alpha}(x)\,dx \quad \text{for } A \subset U,\ A \in \mathfrak{B};$$

μ — an element of $\mathfrak{M}^0$;

E^1 — the set of all expanding mappings of class C^1 of M with the topology induced by the C^1-topology on $C^1(M, M)$, i.e., of all mappings $\varphi\colon M \to M$ of class C^1 for which there exist $a > 0$ and $b > 1$ such that

$$\|(d\varphi^n)(\mathfrak{a})\| \geqslant ab^n \|\mathfrak{a}\| \quad \text{for } \mathfrak{a} \in T(M) \text{ and } n \in N;$$

$\deg\varphi$ — the degree of $\varphi \in E^1$, i.e., $\operatorname{card}\varphi^{-1}(x)$ independent of choice of $x \in M$, since φ is a covering;

$D\varphi$ — the positive continuous function on M such that if $\varphi \in E^1$ is injective on $A \in \mathfrak{B}$, then

$$\mu(\varphi(A)) = \int_A D\varphi\, d\mu; \tag{1}$$

U_φ — the mapping of $C^0(M, R)$ into itself defined in the following way for $\varphi \in E^1$:

$$U_\varphi(f)(x) = \sum_{\bar{x} \in \varphi^{-1}(x)} f(\bar{x})\big(D\varphi(\bar{x})\big)^{-1} \quad \text{for } x \in M,\ f \in C^0(M, R);$$

$|\cdot|$ — the natural norm on R^m.

LEMMA. *Let $\varphi, \psi \in E^1$. Then:*

(a) $D(\psi\circ\varphi) = D\psi\circ\varphi D\varphi$.

(b) *If (U_1, a_1) and (U_2, a_2) are two charts in M such that $\varphi(U_1)\subset U_2$, then*

$$D\varphi\circ a_1^{-1} = \frac{g_{\mu,a_2}\circ a_2\circ\varphi\circ a_1^{-1}\left|\det\left(d(a_2\circ\varphi\circ a_1^{-1})\right)\right|}{g_{\mu,a_1}}.$$

(c) $E^1\ni\varphi\to D\varphi\in C^0(M,R)$ *is continuous.*

(d) $\nu\in\mathfrak{M}^0$ *is φ-invariant iff $U_\varphi(g) = g$, where g is the continuous density of ν with respect to μ.*

(e) E^1 *is open in the C^1-topology and $E^1\ni\varphi\to\deg\varphi\in N$ is locally constant.*

(f) $E^1\ni\varphi\to U_\varphi(1)\in C^0(M,R)$ *is continuous.*

Proof. (a) follows easily from the definition of $D\varphi$.

For the proof of (b) we represent both sides of (1) by means of the integrals with respect to Lebesgue measure and use the theorem on integration by substitution.

(c) follows from (b).

In order to prove (d), it suffices to show that

$$\int_{\varphi^{-1}(A)} g\,d\mu = \int_A U_\varphi(g)\,d\mu \quad \text{for } A\in\mathfrak{B}. \tag{2}$$

For this purpose we may assume that $\operatorname{diam}A$ is sufficiently small. Then (2) follows from the fact that φ is a covering and from (1).

(e) is well known (see [4]).

For the proof of (f), in view of (c) it is sufficient to show that, given $\varphi_0\in E^1$, there is a $\delta_0>0$ such that for each $\delta\in\,]0,\delta_0]$ there exists a neighbourhood $\mathfrak{U}_\delta$ of φ_0 in E^1 such that if $\varphi\in\mathfrak{U}_\delta$, $x\in M$ and $\varphi_0^{-1}(x) = \{x_1^0,\ldots,x_{\deg\varphi_0}^0\}$, then $\deg\varphi = \deg\varphi_0$ and $\varrho(x_i,x_i^0)<\delta$ for $i = 1,\ldots,\deg\varphi_0$, where $\varphi^{-1}(x) = \{x_1,\ldots,x_{\deg\varphi}\}$. For this purpose let us remark that since φ_0 is a covering and since M is compact, there exists a $\delta_0>0$ such that if $x\in M$ and $\varphi_0^{-1}(x) = \{x_1^0,\ldots,x_{\deg\varphi_0}^0\}$, then

$$K(x_1^0,\delta_0),\ldots,K(x_{\deg\varphi_0}^0,\delta_0) \text{ are pairwise disjoint.} \tag{3}$$

Moreover, it is easy to see that there exist a neighbourhood $\mathfrak{U}_0$ of φ_0 in E^1 and $c>0$ such that

$$\|(d\varphi)(\mathfrak{a})\|\geqslant c\|\mathfrak{a}\| \quad \text{for } \varphi\in\mathfrak{U}_0,\ \mathfrak{a}\in T(M). \tag{4}$$

Since any expanding mapping has the property of lifting the curves, (4) implies

$$\varphi\left(K(x,\delta)\right)\supset K\left(\varphi(x),c\delta\right) \quad \text{for } \varphi\in\mathfrak{U}_0,\ x\in M,\ \delta>0. \tag{5}$$

By (e) we may assume that $\deg\varphi = \deg\varphi_0$ for $\varphi \in \mathfrak{U}_0$. Now, let $\delta \in]0, \delta_0]$. Then there exists a neighbourhood $\mathfrak{U}_\delta$ of φ_0 in E^1 contained in $\mathfrak{U}_0$ such that

$$(6) \qquad \varrho\big(\varphi(x), \varphi_0(x)\big) < c\delta \quad \text{for } \varphi \in \mathfrak{U}_\delta,\ x \in M.$$

It follows from (3), (5) and (6) that $\mathfrak{U}_\delta$ has the required property. This completes the proof of the lemma.

The main result of the paper is the following

THEOREM. *The set A of all $\varphi \in E^1$ for which there exists a φ-invariant $\nu \in \mathfrak{M}^0$ is of the first category in E^1.*

Proof. First we prove that

$$(7) \qquad A \subset \bigcup_{k=1}^{\infty} A_k,$$

where A_k ($k \in N$) is the set of all $\varphi \in E^1$ such that

$$(8) \qquad k^{-1} \leqslant U_{\varphi^n}(1) \leqslant k \quad \text{for } n \in N.$$

For this purpose let $\varphi \in A$. Then, by Lemma (d) we obtain

$$(9) \qquad \inf g(\sup g)^{-1} \leqslant U_{\varphi^n}(1) \leqslant \sup g(\inf g)^{-1} \quad \text{for } n \in N,$$

where g is the continuous density of a φ-invariant $\nu \in \mathfrak{M}^0$ with respect to μ. Formula (9) implies (8) for a certain $k \in N$. Therefore, it suffices to prove that, for each $k \in N$,

(10) A_k is a closed and boundary set in E^1.

The first part of (10) follows from the continuity of superposition in $C^1(M, M)$ and from Lemma (f). For the proof of the second part, $k \in N$, $\varphi \in A_k$ and $x_0 \in M$ will be fixed from now on. Then, by Lemma (e), it suffices to show that there exists a sequence (φ_n) in $C^1(M, M)$ such that

(11) $\varphi_n \to \varphi$ in the C^1-topology;

(12) $U_{\varphi_n^n}(1)(x_0) \to +\infty$.

To do this, for $i = 1, \ldots, s$ let (U_i, a_i) and (V_i, β_i) be charts in M, let $Z_i \subset U_i$ be a compact set such that $a_i(Z_i)$ is a ball in R^m ($m = \dim M$), and let

$$\varphi(Z_i) \subset V_i \quad \text{and} \quad \bigcup_{i=1}^{s} \operatorname{int} Z_i = M.$$

Then there exists an L such that, for $i = 1, \ldots, s$,

(13) $\beta_i \circ \varphi \circ a_i^{-1} | a_i(Z_i)$ satisfies the Lipschitz condition with the constant L.

Further, let f: $]-1, 1[\to[0, 1]$ be a function of class C^1 such that

$$f(x) = \begin{cases} 0 & \text{for } 2^{-1} \leqslant |x| < 1, \\ 1 & \text{for } |x| \leqslant 3^{-1}, \end{cases} \tag{14}$$

and let (ε_n) be a sequence of real numbers such that

$$0 < \varepsilon_n < 1, \quad \varepsilon_n \to 0, \quad (1-\varepsilon_n)^{-n} \to +\infty. \tag{15}$$

Now, for each sufficiently large $n \in N$ we define $\varphi_n \in C^1(M, M)$ as follows:

Let $n \in N$ be fixed and let

$$\bigcup_{k=1}^{n} \varphi^{-k}(x_0) = \bigcup_{i=1}^{s} W_{n,i},$$

where $W_{n,i} \subset \operatorname{int} Z_i$ for $i = 1, \ldots, s$, and $W_{n,1}, \ldots, W_{n,s}$ are pairwise disjoint. From now on, for $\bar{x} \in W_{n,i}$, $\bar{y}$ will denote $a_i(\bar{x})$ $(i = 1, \ldots, s)$. Then there exists a $\delta_n > 0$ such that

(16) $K(\bar{y}, \delta_n) \subset a_i(Z_i)$ for $\bar{x} \in W_{n,i}$ and $i = 1, \ldots, s$;

(17) the sets $a_i^{-1}(K(\bar{y}, \delta_n))$ are pairwise disjoint, where $\bar{x} \in W_{n,i}$ and $i = 1, \ldots, s$.

Now we define φ_n modifying φ only at the points of the sets from (17). For this purpose let us remark that, in view of (15), if $n \in N$ is sufficiently large and $\bar{x} \in W_{n,i}$ for $i = 1, \ldots, s$, then for each $y \in K(\bar{y}, \delta_n)$

$$(\beta_i \circ \varphi \circ a_i^{-1})(y) - \\ - \varepsilon_n f(\delta_n^{-2} |y-\bar{y}|^2)\big((\beta_i \circ \varphi \circ a_i^{-1})(y) - (\beta_i \circ \varphi \circ a_i^{-1})(\bar{y})\big) \in \beta_i(V_i). \tag{18}$$

Therefore, in view of (17), for such an n there exists a φ_n: $M \to M$ such that $(\beta_i \circ \varphi_n \circ a_i^{-1})(y)$ is equal to the left-hand side of (18) for $y \in K(\bar{y}, \delta_n)$, where $\bar{x} \in W_{n,i}$, $i = 1, \ldots, s$, and such that at each point x,

$$x \notin \bigcup_{i=1}^{s} \bigcup_{\bar{x} \in W_{n,i}} a_i^{-1}(K(\bar{y}, \delta_n)), \tag{19}$$

φ_n and φ attain the same values. Then, by formula (14) and Lemma (b), $\varphi_n \in C^1(M, M)$ and

$$\varphi_n(\bar{x}) = \varphi(\bar{x}), \quad D\varphi_n(\bar{x}) = (1-\varepsilon_n)^m D\varphi(\bar{x}) \quad \text{for } \bar{x} \in \bigcup_{k=1}^{n} \varphi^{-k}(x_0). \tag{20}$$

For the proof of (11) let us remark that, from the definition of φ_n and (13), for $y \in K(\bar{y}, \delta_n)$, $\bar{x} \in W_{n,i}$ and $i = 1, \ldots, s$ we obtain

$$|(\beta_i \circ \varphi \circ a_i^{-1})(y) - (\beta_i \circ \varphi_n \circ a_i^{-1})(y)| \leqslant 2\varepsilon_n d_i, \tag{21}$$

$$|d(\beta_i \circ \varphi \circ a_i^{-1})(y) - d(\beta_i \circ \varphi_n \circ a_i^{-1})(y)| \leqslant \varepsilon_n e_i + 2\varepsilon_n L d, \tag{22}$$

where

$$d_i = \sup_{y \in a_i(Z_i)} |(\beta_i \circ \varphi \circ a_i^{-1})(y)|, \qquad e_i = \sup_{y \in a_i(Z_i)} |d(\beta_i \circ \varphi \circ a_i^{-1})(y)|$$

and

$$d = \sup_{|x|<1} |f'(x)|.$$

Moreover, if (19) holds, then there exists a neighbourhood of x in M such that φ_n and φ attain the same values at each point belonging to it. Hence, by (15), (21) and (22) we obtain (11).

For the proof of (12) let us observe that (11), Lemma (a), (e) and (20) imply that, for sufficiently large $n \in N$,

$$U_{\varphi_n^n}(1)(x_0) = (1-\varepsilon_n)^{-mn} U_{\varphi^n}(1)(x_0).$$

Hence, by (15) and the definition of A_k, we get (12).

Thus the proof of the theorem is completed.

Since $C^1(M, M)$ is a Baire space, by Lemma (e) and the Theorem we have $A \neq \emptyset$ provided that $E^1 \neq \emptyset$.

Now we state without proof a theorem which completes the results of [2] and [3]:

Let φ be an expanding mapping of class C^2 of a compact, connected differentiable manifold and let ν be a normalized φ-invariant measure of class C^1. Then the natural extension of the dynamical system (ν, φ) to an automorphism is a Bernoulli shift.

The proof is based on [1]. Moreover, it turns out that in such a way one can obtain all Bernoulli shifts with positive finite entropy.

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