COLLOQUIUM MATHEMATICUM

VOL. XV 1966 FASC. 2

ON STEMS

BY

R. DUDA (WROCŁAW)

The paper consists of two parts. In the first one we introduce a notion of a stem, i.e. of an ordered set with a special kind of order, and show what are its topological and metric equivalents. And in the second we turn to the question of a reconstruction, by means of functions of dimension, of that order from a proper topology acting on a stem.

1. An ordered set $(X, \prec)$ is called a *stem* if the relation $\prec$ partially orders X, and if

(i) there exists in X an element a which precedes any other element of X (i.e. $a \prec x$ for all $x \in X$),

(ii) for every $x \in X$, there exist only finitely many elements of X which precede x,

(iii) for every $x \in X$, all elements of X which precede x are linearly ordered.

The point a is called a *germ* of the stem X. Obviously, any stem contains one germ only.

There are several natural examples of stems. Such is, for instance, the set of natural numbers with the order $\leqslant$ and 0 as a germ. And if we consider a Cantor fan, i.e. the union of all rectilinear segments $L(\tau)$ connecting points $(\tau, 0)$ of a Cantor ternary set with a point, say, $(\frac{1}{2}, 1)$, then any subset X of $\bigcup L(\tau)$ containing the point $(\frac{1}{2}, 1)$ and meeting each segment $L(\tau)$ in a finite set may be regarded as a stem if we agree to the definition that point $x \in X$ precedes point $y \in X$, $x \prec y$, if and only if both belong to the same segment $L(\tau)$ and ordinate of x is not greater than that of y. In particular, a vast class of biconnected sets consists of stems (cf. [2]).

Also, if we consider the greatest lower bound of any two elements of a stem X, x and y say, to be the greatest $z \in X$ with the property $z \prec x$ and $z \prec y$, then stems are simple examples of semilattices in the sense of G. Birkhoff [1]. In particular, each stem is a directed set, because the inverse relation $\succ$ directs it (for relevant definitions see also [4]).

Now we shall introduce into every stem a topology — called in the sequel a *proper* one — which depends entirely on the order structure of X and reflects it completely. Namely, for a given $b \in X$ we define the least open neighbourhood U_b of b to be the set of all those x, including a and b, which lie between a and b. By virtue of (i) and (ii) any such neighbourhood is a finite set, and by (iii) the intersection of any two of these neighbourhoods is again a neighbourhood of some point of X. Open sets in X are the unions of these neighbourhoods. In particular, the germ a forms an open set, but the only closed set to which it belongs is the whole X. And no other point of X has that property. Obviously, the proper topology turns X into a T_0-space.

If X is a topological space, then — as one can easily see — the topology acting on X is proper if the following three conditions hold true:

a) there exists a point a which belongs to every non-void and open subset of X,

b) there exists, for every point x of X, the least neighbourhood of x which is a finite set,

and, denoting the least neighbourhood of x by U_x,

c) if $x \in X$, $b \in U_x$, and $c \in U_x$, then either $b \in U_c$ or $c \in U_b$.

In fact, these three conditions correspond to conditions (i)-(iii) imposed on order structure of stems if we introduce an order into X by the definition

$$x \preccurlyeq y \quad \text{if and only if} \quad x \in U_y.$$

Since both definitions of a stem X, that with the help of an order and that given by a proper topology, are equivalent, we shall not distinguish between the two in the sequel.

Now let b and c be two elements of a stem X. By virtue of the definition of an order in X, there exists an element $e \in X$ which precedes both b and c, and is the greatest with this property. Furthermore, all elements of X which lie between e and b may be arranged in a finite sequence $e = x_0 \preccurlyeq x_1 \preccurlyeq \dots \preccurlyeq x_k = b$, and similarly, all elements of X which lie between e and c may be arranged in a finite sequence $e = y_0 \preccurlyeq \preccurlyeq y_1 \preccurlyeq \dots \preccurlyeq y_1 = c$. Let us denote the set

$$\{b = x_k, x_{k-1}, \dots, x_1, x_0 = e = y_0, y_1, \dots, y_1 = c\}$$

by $L(b, c)$. The pair (k, l) of natural numbers will be denoted by $n(b, c)$ and called *relative number of knots*, and the sum $k+l$ will be denoted by $n^*(b, c)$ and called *absolute number of knots* between b and c.

Hence the absolute number of knots tells us how many "knots" we have on the way from b to c, and the relative number of knots is more precise to say how many down and how many up.

Notice that if X is a stem with a germ a provided with a proper topology, then

$$L(a, x) = U_x \quad \text{for every} \quad x \in X.$$

Moreover, if d is the greatest element of X preceding two given elements of X, b and c, then

$$L(b, c) = [(U_b - U_b \cap U_c) \cup \{d\}] \cup [(U_c - U_b \cap U_c) \cup \{d\}]. \tag{1}$$

Consequently,

$$n(b, c) = [n^*(a, b) - n^*(a, d) + 1, n^*(a, c) - n^*(a, d) + 1]. \tag{2}$$

In fact, the first sommand of the right-hand side of (1) consists of all those elements of $L(b, c)$ that precede b, and their number is equal to the number of elements in U_b minus the number of elements in $U_b \cap U_c$ plus 1, i.e. to $n^*(a, b) - n^*(a, d) + 1$. Similarly for the second sommand.

Notice also that elements b and c are comparable if and only if $n(b, c) = (k, l)$ and either $k = 0$ or $l = 0$; if $k = 0$, then $b \preceq c$, and if $l = 0$, then $c \preceq b$. Consequently,

$$n(b, c) = (0, 0) \text{ if and only if } n^*(b, c) = 0 \text{ if and only if } b = c. \tag{3}$$

If we have three collinear points $x \preceq z \preceq y$, then obviously $L(x, y) = L(x, z) \cup L(z, y)$, and consequently,

$$n^*(x, y) = n^*(x, z) + n^*(z, y). \tag{4}$$

It should be equally obvious that if, for given x and y, an element e is the greatest one preceding both x and y, then

$$n^*(x, y) = n^*(x, e) + n^*(e, y). \tag{5}$$

Finally, let us remark that if X is a stem, then the function $n(x, y)$ ranging over $X \times X$ completely determines the order structure in X. For instance, germ a is the only point of X having the property $n(a, y) = (0, 1)$ for any $y \in X$. And if it happens that we do know what a point a is the germ of a stem X, then the function $n^*(x, y)$ already suffices to re-read the order structure in X, In fact, with the help of function $n^*(a, x)$ we can find the set $L(a, b)$ for every $b \in X$, because

$$L(a, b) = \{x: n^*(a, x) + n^*(x, b) = n^*(a, b), x \in X\}.$$

And now, given two elements b and c, we simply check the number of elements in the sets $L(a, b) - L(a, c)$ and $L(a, c) - L(b, c)$. If it is equal, respectively, to k and l, then we have $n(b, c) = (k, l)$. In particular, $b \preceq c$ if and only if $L(a, b) \subset L(a, c)$.

We say that a metric space (X, ϱ) is *integer-convex* if

(i) ϱ is a metric,

(ii) for all $x \in X$ and $y \in X$, $\varrho(x, y)$ is a non-negative integer,

(iii) for every pair x and y of elements of X and every integer $0 < k < \varrho(x, y)$ there exists exactly one point $z \in X$ such that

$$\varrho(x, z) = k \quad \text{and} \quad \varrho(z, y) = \varrho(x, y) - k.$$

The set of natural numbers with the metric $\varrho(m, n) = |m-n|$ is a simple example of an integer-convex space.

More generally,

THEOREM 1. *Every stem X is an integer-convex space with the metric*

$$\varrho(b, c) = n^*(b, c), \quad b \in X, c \in X.$$

Proof [1]. We have to check first that n^* is a metric. By virtue of (3) there is $n^*(b, c) = 0$ if and only if $b = c$. And since symmetry condition

$$n^*(b, c) = n^*(c, b) \quad \text{for all} \quad b \in X \text{ and } c \in X$$

is equally obvious, it remains to see the triangle condition only.

Let b, c, and d be three arbitrary elements of X. Denote by e_1, e_2, and e_3 the greatest elements of X preceding, respectively, e_1 — both b and d, e_2 — both c and d, and e_3 — both b and c. Since both e_1 and e_2 precede d, then there is either $e_1 \preccurlyeq e_2$ or $e_2 \preccurlyeq e_1$. For the reason of symmetry suppose that $e_1 \preccurlyeq e_2$. Then, by virtue of $e_1 \preccurlyeq b$ and $e_2 \preccurlyeq c$, the element e_1 precedes both b and c, and therefore, in view of the definition of e_3, there is also $e_1 \preccurlyeq e_3$.

In view of (5) we have then

$$n^*(b, c) = n^*(b, e_3) + n^*(e_3, c), \tag{6}$$

$$n^*(b, d) = n^*(b, e_1) + n^*(e_1, d), \tag{7}$$

$$n^*(d, c) = n^*(d, e_2) + n^*(e_2, c). \tag{8}$$

And, since $e_1 \preccurlyeq e_3 \preccurlyeq b$, by (4)

$$n^*(b_1, e_1) \geqslant n^*(b, e_3). \tag{9}$$

Consider now two particular cases.

I. $e_2 \preccurlyeq e_3$. In that case we have $c \preccurlyeq e_2 \preccurlyeq e_3$, whence we infer by (4) that

$$n^*(e_2, c) \geqslant n^*(e_3, c). \tag{10}$$

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[1] The author is indebted to Dr. L. W. Nitka for some improvements of the original proof.

Replacing now two right-hand sommands of (6) by $n^*(b, e_1)$ and $n^*(e_2, c)$, respectively, we obtain by virtue of (9) and (10) the inequality

$$n^*(b, c) \leqslant n^*(b, e_1) + n^*(e_2, c),$$

the right-hand side of which is obviously not greater than the sum of (7) and (8).

II. $e_3 \prec e_2$. In that case using the fact that $e_1 \prec e_3 \prec e_2 \prec d$, we infer by (4) that

$$n^*(e_1, d) \geqslant n^*(e_3, e_2). \tag{11}$$

And since $e_3 \prec e_2 \prec c$, by applying again (4), we have

$$n^*(e_3, e_2) + n^*(e_2, c) = n^*(e_3, c). \tag{12}$$

Replacing now two right-hand sommands of (6) by $n^*(b, e_1)$ and $n^*(e_3, e_2) + n^*(e_2, c)$, respectively, we obtain by (9) and (12) the inequality

$$n^*(b, c) \leqslant n^*(b, e_1) + n^*(e_3, e_2) + n^*(e_2, c),$$

whence by an application of (11) we get

$$n^*(b, c) \leqslant n^*(b, e_1) + n^*(e_1, d) + n^*(e_2, c),$$

and it suffices only to add to the right-hand side of the last inequality the sommand $n^*(d, e_2)$ in order to get the sum of (7) and (8).

Thus we have proved that the function n^* is a metric on X. It obviously satisfies (ii) by its very definition, and since for every two elements $b \in X$ and $c \in X$ the set $L(b, c)$ is uniquely determined, it satisfies (iii) too. Hence the proof of Theorem 1 is completed.

REMARK. A part of Theorem 1 holds true in a more general case. Namely, if X is a semi-lattice with greatest lower bound $x \wedge y$ and with a norm $\|x\|$ satisfying the condition:

$$x \prec y \text{ implies } \|x\| \prec \|y\|,$$

then defining

$$\varrho(x, y) = \|x\| + \|y\| - 2\|x \wedge y\|$$

we easily check that the quite analogical proof to that of Theorem 1 works to the end that ϱ is a metric.

However, since this generalization presents no difficulty, in view of the clearness of the paper we have consciously restricted ourselves to a particular case of an integer-convex space.

Any integer-convex space may be turned into stem, and in fact usually into many stems. For if $Q(x, y)$ denotes a segment in an integer-convex space between its points x and y, then we have the following theorem:

THEOREM 2. *Let X be an integer-convex space and a any point of X. Then putting*

$$U_x = Q(a, x) \quad \text{for every} \quad x \in X,$$

we get a stem with a germ a.

Proof. We shall check conditions a), b), and c).

a) Obviously, $a \in U_x$ for all $x \in X$.

b) Since every segment $Q(a, x)$ is, by the definition of an integer-convex space, a finite set, U_x is finite for all $x \in X$.

c) Finally, if $x \in X$, and $b \in U_x$ and $c \in U_x$ are two distinct points of U_x, then by the definition of an integer-convex space and the definition of U_x there is either $\varrho(a, b) < \varrho(a, c)$ or $\varrho(a, c) < \varrho(a, b)$. In the first case we have $b \in U_c$ and in the second $c \in U_b$.

2. For a reconstruction of the order structure from a proper topology acting on a stem we shall use functions of *γ-dimension*. Let us recall them (see [3]).

Denoting by Γ the set of all sequences consisting of 0's and 1's only, we assign to any $\gamma \in \Gamma$ a function of γ-dimension as follows:

(i) if $X = \emptyset$, then $\gamma\text{-dim}\, X = -1$,

(ii) if $X \neq \emptyset$, then $\gamma\text{-dim}\, X = \sup\limits_{Y \in X/\gamma_1} \gamma\text{-dim}_Y X$,

where X/γ_1 is X itself if $\gamma_1 = 0$ or the family of all closed subsets of X if $\gamma_1 = 1$,

(iii) $\gamma\text{-dim}_Y X \leqslant n$ means that for every neighbourhood U of Y there exists a neighbourhood V of Y such that $V \subset U$ and

$$(\gamma_2, \gamma_3, \ldots)\text{-dim}\, \mathrm{Fr}(V) \leqslant n-1.$$

A particular sequence consisting of 0's only will be denoted by a_0. Obviously, the definition of a_0-dim is precisely that of Menger's inductive dimension ind.

To find γ-dimension for a finite set we shall apply procedure of [3] consisting in finding out those γ-sequences of a given set, which have the maximal length. By a *γ-sequence* of X we mean any triple sequence $\{H_j, U_j, F_j\}_{j=1,2,\ldots,k}$ such that

(a) H_1 is an element of X/γ_1, U_1 is the least open subset of X containing H_1, and F_1 is the boundary of U_1;

(b) if $j \geqslant 1$, then H_{j+1} is an element of F_j/γ_{j+1}, U_{j+1} is the least open subset of F_j containing H_{j+1}, and F_{j+1} is the boundary of U_{j+1} relative to $F_j (F_j/\gamma_{j+1}$ is equal to F_j for $\gamma_{j+1} = 0$ or to the family of its all closed subsets for $\gamma_{j+1} = 1)$;

(c) $H_j \neq \emptyset \neq U_j$ for all $j = 1, 2, \ldots, k$, and $F_k = \emptyset$.

LEMMA 1. *If X is a stem with a germ a, and c is any point of X, then* $n^*(a, c) = a_0\text{-dim}\, U_c$.

Proof. Let $U_c = \{a = x_0, x_1, \ldots, x_k = c\}$. The only non-void subsets of U_c in the relative topology are the sets $\{x_0, x_1, \ldots, x_i\}$, where $i = 0, 1, \ldots, k$.

We shall construct an a_0-sequence of a maximal length. The best choice for H_1 is x_0, for we have then

$$H_1 = \{x_0\}, \quad U_1 = \{x_0\}, \quad F_1 = \{x_1, x_2, \ldots, x_k\},$$

and having made any other choice for H_1 we are left with a proper subset of F_1.

It is easy to proceed further on. Namely,

$$H_{i+1} = \{x_i\}, \quad U_{i+1} = \{x_i\}, \quad F_{i+1} = \{x_{i+1}, x_{i+2}, \ldots, x_k\}$$

for $i = 1, 2, \ldots, k-1$, and

$$H_{k+1} = \{x_k\}, \quad U_{k+1} = \{x_k\}, \quad F_{k+1} = \emptyset.$$

Hence by Corollary 1 of [3] there is

$$a_0\text{-dim}\, U_c = k,$$

and our lemma follows.

THEOREM 3. *Let X be a stem with a proper topology. If $b \in X$ and $c \in X$, then*

$$a_0\text{-dim}\, U_b = n^*(a, b), \quad a_0\text{-dim}\, U_c = n^*(a, c),$$

and

$$n(b, c) = [n^*(a, b) - k + 1, n^*(a, c) - k + 1],$$

where a is a germ of X, U_x is the least neighbourhood of x, and k is the greatest natural number with the property that there exists $d \in U_b \cap U_c$ such that $a_0\text{-dim}\, U_d = k$.

Proof. The first two equalities follow by Lemma 1. In particular, the germ a is the only element z of X for which $a_0\text{-dim}\, U_z = 0$.

Now let d be the point of $U_b \cap U_c$ such that $k = a_0\text{-dim}\, U_d$ is the greatest natural number. Then d is the greatest element of x preceding b and c, and to obtain the second equality it suffices to apply (2).

LEMMA 2. *Let Y be a stem consisting of two finite linearly ordered sets $b_0 \prec b_1 \prec \ldots \prec b_k$ and $c_0 \prec c_1 \prec \ldots \prec c_{k+s}$, where $k \geqslant 1$, $s \geqslant 0$, $b_0 = c_0$, and let no b_i be comparable with any c_j for $i \geqslant 1$ and $j \geqslant 1$.*

If Y is provided with a proper topology, then

$$\gamma\text{-dim}\, Y \leqslant k + s \quad \textit{for all} \quad \gamma \in \Gamma$$

and there exists unique (up to the first $k+s$ places) $\beta \in \Gamma$ such that

$$\beta \neq \alpha_0 \quad \text{and} \quad \beta\text{-dim}\, Y = k+s.$$

More precisely, $\beta_1 = 1$ *and* $\beta_i = 0$ *for* $i = 2, 3, \ldots, k+s$.

Proof. We shall construct step by step a γ-sequence of a maximal length. The best choice for H_1 is either any point b_i or a closed subset of $\{b_0, b_1, \ldots, b_k\}$. In the first case we have $\gamma_1 = 0$ and

$$H_1 = \{b_i\}, \quad U_1 = \{b_0, b_1, \ldots, b_i\}, \quad F_1 = \{b_{i+1}, \ldots, b_k\} \cup \{c_1, c_2, \ldots, c_{k+s}\}$$

Since both sommands of F_1 are obviously non-void and closed-open subsets of F_1, then

$$\gamma\text{-dim} F_1 = \max[\gamma\text{-dim}\{b_{i+1}, \ldots, b_k\},\ \gamma\text{-dim}\{c_1, \ldots, c_{k+s}\}]$$

for all $\gamma \in \Gamma$. As $\{b_{i+1}, \ldots, b_k\}$ is homeomorphic to the subset $\{c_1, \ldots, c_{k-i}\}$ of $\{c_1, \ldots, c_{k+s}\}$, we have

$$\gamma\text{-dim}\{b_{i+1}, \ldots, b_k\} \leqslant \gamma\text{-dim}\{c_1, \ldots, c_{k+s}\},$$

and therefore we may neglect the first sommand. Hence, practically, we are left with $\{c_1, \ldots, c_{k+s}\}$.

In the second case we have $\gamma_1 = 1$ and $H_1 = H$, where $H = \bar{H} \subset \{b_0, \ldots, b_k\}$, $U_1 = \{b_0, \ldots, b_k\}$ and $F_1 = \{c_1, \ldots, c_{k+s}\}$.

Having made any other choice for H_1 (i.e. choosing a point c_j or a closed subset of $\{c_1, \ldots, c_{k+s}\}$) we are left with a proper subset of $\{c_1, \ldots, c_{k+s}\}$ with adjoined to it set $\{b_1, \ldots, b_k\}$ which — as we just have seen — has the less γ-dimensional value that $\{c_1, \ldots, c_{k+s}\}$.

In both cases the best choice for H_2 is c_1, because we have then $H_2 = \{c_1\}$, $U_2 = \{c_1\}$, and $F_2 = \{c_2, c_3, \ldots, c_{k+s}\}$. Having made any other choice, we are left with a proper subset of F_2. Since c_1 is not closed in F_1, then we have $\gamma_2 = 0$.

Now it is easy to proceed further on. Namely, the best possible choice in any subsequent step is the first element relative to order, which gives us

$$H_{i+1} = \{c_i\}, \quad U_{i+1} = \{c_i\}, \quad F_{i+1} = \{c_{i+1}, \ldots, c_{k+s}\}, \quad \text{and} \quad \gamma_{i+1} = 0$$

for all $i = 1, 2, \ldots, k+s-1$, and

$$H_{k+s+1} = \{c_{k+s}\}, \quad U_{k+s+1} = \{c_{k+s}\},$$

$$F_{k+s+1} = \emptyset, \quad \text{and} \quad \gamma_{k+s+1} = 0 \text{ or } 1.$$

By Corollary 1 of [3] we have then $\gamma\text{-dim}\, Y = k+s$ for the best $\gamma \in \Gamma$, and since γ_1 may be equal to 1, the Lemma is proved.

THEOREM 4. *Let X be a stem, and b and c two incomparable points of X such that $n^*(a, b) \leqslant n^*(a, c)$, where a denotes the germ of X. If V is the least neighbourhood of b and c, then*

$$\gamma\text{-}\dim V \leqslant n^*(a, c) \quad \textit{for all} \quad \gamma \in \Gamma,$$

and there exists a unique (up to the first $n^*(a, c)$ places) $\beta \in \Gamma$ such that

$$\beta \neq a_0 \quad \textit{and} \quad \beta\text{-}\dim V = n^*(a, c).$$

More precisely, $\beta_i = 0$ for all $i = 1, 2, \ldots, n^*(a, c)$ except for the greatest k with the property that there exists $x \in X$ such that $x \precsim b$, $x \precsim c$, and $n^*(a, x) = k$.

Proof. The set V is a union of the least neighbourhoods of b and c; and therefore can be written in the form

$$a = a_0 \precsim a_1 \precsim \ldots \precsim a_k \begin{cases} a_k = b_k \precsim b_{k+1} \precsim \ldots \precsim b_{k+l} = b, \\ a_k = c_k \precsim c_{k+1} \precsim \ldots \precsim c_{k+l+m} = c. \end{cases}$$

We shall construct a γ-sequence of a maximal length. The best point to start with is $a = a_0$, because we have then

$$H_1 = \{a_0\}, \quad U_1 = \{a_0\}, \quad F_1 = V - \{a_0\}.$$

Having made any other choice for H_1 we are left with a proper, subset of F_1. Since a_0 is not closed in Y, we get $\gamma_1 = 0$.

In analogical way we may proceed further on. Namely, the best possible choice for H_{i+1}, where $i = 1, 2, \ldots, k-1$, is the least element left, that is

$$H_{i+1} = \{a_i\}, \quad U_{i+1} = \{a_i\}, \quad F_{i+1} = F_i - \{a_i\}.$$

All the time we have $\gamma_i = 0$.

At last we come to $F_k = V - \{a_0, \ldots, a_{k-1}\}$, which is a set Y of Lemma 2. Applying it, we complete the proof.

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INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 25. 5. 1965