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A CONTROL ON THE SET WHERE A GREEN'S FUNCTION VANISHES

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1. Introduction. In this paper we consider the behavior of a Green's function $g(x,t;y,s)$ associated with the nondivergence form parabolic operator

$$L = \sum_{i,j=1}^{n} a_{ij}(x,t) D^2_{x_i x_j} - D_t$$

and with the cylinder $B \times \mathbf{R}$. Here B is a ball contained in $\mathbf{R}^n$, $\mathbf{R}$ is the real line, x belongs to $\mathbf{R}^n$, and t belongs to $\mathbf{R}$. About the matrix $a(x,t) \equiv (a_{ij}(x,t))$, we assume it is symmetric and there exists λ, $0 < \lambda \le 1$, such that for all $x \in \mathbf{R}^n$, $\xi \in \mathbf{R}^n$, and $t \in \mathbf{R}$

$$\lambda |\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i \xi_j \le \frac{1}{\lambda}|\xi|^2.$$

The parameter λ will be referred to as the *parameter of parabolicity* of L. We will assume the coefficients $a_{ij}(x,t)$ are smooth functions, but it is only in terms of the quantities λ and n that we wish to describe a control of the zero set of $g(x,t;y,s)$ in the adjoint variable (y,s), $y \in \mathbf{R}^n$, $s \in \mathbf{R}$. Specifically, we prove the existence of positive numbers M_λ and c_λ, depending only on λ and n, such that for every measurable set $\Gamma \subset C_1 \equiv \{(y,s) : |y_i| \le 1/2, i = 1,\ldots,n, 0 \le s \le 1\}$ and for any B with center the origin and containing $Q_2 \equiv \{y : |y_i| \le 1, i = 1,\ldots,n\}$, we have

$$\int_\Gamma g(0,2;y,s)\,dy\,ds \ge c_\lambda |\Gamma|^{M_\lambda}, \tag{1.1}$$

where $|\Gamma|$ denotes the Lebesgue measure of Γ. This estimate is a refinement of a related result due to N. V. Krylov and M. V. Safanov. In [7] they announce a lower bound for the above integral with $c_\lambda|\Gamma|^{M_\lambda}$ replaced by a function $\phi(|\Gamma|)$ satisfying $\phi(t) > 0$ for $t > 0$.

In case the coefficients of the operator L are independent of time, $g(x,t;y,s) \equiv g(x,t-s,y)$ and an estimate like (1.1) holds for each fixed time with Γ any measurable subset of the unit cube in $\mathbf{R}^n$. While this last statement is a consequence of (1.1) and Harnack's inequality [8], it is also a consequence of the earlier work [4]. However, the estimate corresponding to (1.1) for fixed times, say for the function $g(0,2;y,0)$, and Γ any measurable subset of the unit cube in $\mathbf{R}^n$, is not possible for parabolic operators with time dependent coefficients. In these cases the measure on $\mathbf{R}^n$ describing solutions of the classical initial value problem can be carried on a set of Lebesgue measure zero [3].

A general outline of the paper is as follows: Section 2 is primarily a preparation for the proof of the main inequality (1.1). However, the one new result of Section 2, Theorem 4, may be of independent interest. It describes a Harnack inequality satisfied by the quotient of a nonnegative solution of the adjoint parabolic equation which vanishes on the lateral boundary and a Green's function corresponding to the same operator.

Section 3 is devoted to the proof of (1.1) and Section 4 discusses applications to a priori bounds for smooth functions $u(x,t)$ vanishing on the parabolic boundary of $B_1 \times (0,1)$. (B_r denotes a ball of radius $r > 0$.) In Section 4 we prove the existence of positive numbers C_λ and ε_λ depending only on λ and n such that for all u described above,

$$\int_0^1 \int_{B_1} |\nabla_x u(x,t)|^{\varepsilon_\lambda}\, dx\, dt + \int_0^1 \int_{B_{1/2}} |D^2_{x_i x_j} u(x,t)|^{\varepsilon_\lambda}\, dx\, dt$$
$$\leq C_\lambda \left[\int_0^1 \int_{B_1} |Lu(x,t)|^{n+1} \right]^{\varepsilon_\lambda/(n+1)}.$$

The elliptic analogue of this estimate was proved by L. C. Evans [2] for the gradient and by Fang-Hua Lin [9] for the second derivatives. (See also [12] for the parabolic case.)

2. Comparison results and a property of adjoint solutions. In this section we first state three results from [5] concerning properties of nonnegative solutions of parabolic equations which vanish on the lateral part of a cylinder. As a consequence we will prove a Harnack inequality for suitable quotients of adjoint solutions (see Definition 1 and Theorem 4).

Let B be a ball in $\mathbf{R}^n$, centered at the origin, and set

$$D_T = B \times (0,T), \qquad D_{\delta,T} = B \times (\delta, T) \quad (\delta > 0).$$

Recall $L \equiv \sum_{i,j=1}^n a_{ij}(x,t) D^2_{x_i x_j} - D_t$.

THEOREM 1. *Let u be a nonnegative solution of $Lu = 0$ in D_T which* *vanishes on $\partial B \times (0,T)$, $T \leq T_1$. Fix a compact $K \subset B$. Then there exists a constant C depending only on λ, n, δ, T_1, the diameter of B and the distance of K to ∂B such that*

$$\sup_{D_{\delta,T}} u \leq C \inf_{K\times(\delta,T)} u.$$

THEOREM 2. *Let u, v be two nonnegative solutions of $Lu = 0$ vanishing on $\partial B \times (0,T)$ with $T \leq T_1$. Fix a compact set $K \subset B$. Then there exists a positive constant C depending only on λ, n, δ, T_1, the diameter of B and the distance of K to ∂B such that*

$$\sup_{D_{\delta,T}} u/v \leq C \inf_{K\times(\delta,T)} u/v.$$

THEOREM 3. *Let u be a nonnegative solution of $Lu = 0$ in D_T vanishing on $\partial B \times (0,T)$ with $T \leq T_1$. Then there exists a positive constant C depending only on λ, n, δ, T_1 and the diameter of B such that*

$$C^{-1}u(0,T') \leq \partial_{v_\varphi} u(\varphi,s) \leq Cu(0,T')$$

for every $(\varphi,s) \in \partial B \times (\delta,T)$ and every T', $\delta \leq T' \leq T$; ∂_{v_φ} denotes the conormal derivative, i.e. $v_\varphi = a(\varphi,s)(N_\varphi)$ where N_φ is the unit inner normal to ∂B at φ.

We now consider adjoint solutions of parabolic operators. We say that $v = v(\xi,\tau)$ is an *adjoint solution* of L in D_T if v satisfies the equation

$$L^*v(\xi,\tau) \equiv D_\tau v(\xi,\tau) + \sum_{i,j=1}^{n} D^2_{\xi_i\xi_j}(a_{ij}(\xi,\tau)v(\xi,\tau)) = 0$$

in D_T.

Let $\widetilde{B}$ be a large ball containing B and denote by $g_{\widetilde{B}}(x,t;\xi,\tau)$ the Green's function for L in the infinite cylinder with cross-section $\widetilde{B}$. As a function of the second set of variables (ξ,τ), $g_{\widetilde{B}}(x,t;\xi,\tau)$ is a nonnegative adjoint solution in the punctured cylinder $\widetilde{B} \times (-\infty,\infty) \setminus \{(x,t)\}$.

DEFINITION 1. If $v = v(\xi,\tau)$ is an adjoint solution in D_T the function

$$w(\xi,\tau) = \frac{v(\xi,\tau)}{g_{\widetilde{B}}(A,R;\xi,\tau)}, \quad A \in \widetilde{B}\setminus\overline{B}, \quad R \geq 2T,$$

is called a *normalized adjoint solution* for L in D_T. Quotients of this type are the analogues of those introduced by P. Baumann [1] in the elliptic case.

THEOREM 4. *Suppose w is a nonnegative normalized adjoint solution for L in D_T, vanishing on $\partial B \times (0,T)$ with $T \leq T_1$. Let (ξ_1,τ_1) and (ξ_2,τ_2) denote two points in D_T with $\tau_2 - \tau_1 \geq \eta > 0$ and $T - \tau_2 \geq \mu > 0$. Then there exists a constant C depending only on λ, n, T_1, η, μ, the diameter* *of B and on* $\min[\operatorname{dist}(\xi_1,\partial B),\operatorname{dist}(\xi_2,\partial B)]$ *such that*

$$w(\xi_1,\tau_1)\le Cw(\xi_2,\tau_2).$$

Remark. We point out that the constant C in Theorem 4 does not depend on the diameter of $\widetilde{B}$ or on the point $(A,R)\in(\widetilde{B}\setminus\overline{B})\times[2T,\infty)$.

Proof. The following representation formula holds for w in D_T:

$$w(\xi,\tau)=\frac{1}{g_{\widetilde{B}}(A,R;\xi,T)}\int_B g_B(x,T;\xi,\tau)g_{\widetilde{B}}(A,R;x,T)w(x,T)\,dx. \tag{2.1}$$

Furthermore, since $g_{\widetilde{B}}(A,R;\cdot,\cdot)$ is an adjoint solution in D_T, we can write

$$\begin{aligned} g_{\widetilde{B}}(A,R;\xi,\tau)=&\int_B g_B(x,T;\xi,\tau)g_{\widetilde{B}}(A,R;x,T)\,dx\\ &+\int_\tau^T\int_{\partial B}\partial_{v_\varphi}g_B(\varphi,s;\xi,\tau)\cdot g_{\widetilde{B}}(A,R;\varphi,s)\,d\varphi\,ds,\end{aligned} \tag{2.2}$$

where ∂_{v_φ} denotes the conormal derivative in the variables (φ,s). If $T_0=\frac{1}{2}(T+\tau_2)$, by Theorem 2, we have

$$\frac{g_B(x,T;\xi_2,\tau_2)}{g_B(x,T;\xi_1,\tau_1)}\sim\frac{g_B(0,T_0;\xi_2,\tau_2)}{g_B(0,T_0;\xi_1,\tau_1)} \tag{2.3}$$

for every $x\in B$, with equivalence constant depending on λ, n, T_1, μ and the diameter of B. On the other hand, using Hopf's Lemma [10] and Theorem 3, we have

$$\begin{aligned} &\int_{\tau_2}^T\int_{\partial B}\partial_{v_\varphi}g_B(\varphi,s;\xi_2,\tau_2)g_{\widetilde{B}}(A,R;\varphi,s)\,d\varphi\,ds\\ &\le C\int_{\tau_2}^T\int_{\partial B}g_B(0,T_0;\xi_2,\tau_2)g_{\widetilde{B}}(A,R;\varphi,s)\,d\varphi\,ds\\ &\le C\frac{g_B(0,T_0;\xi_2,\tau_2)}{g_B(0,T_0;\xi_1,\tau_1)}\int_{\tau_2}^T\int_{\partial B}\partial_{v_\varphi}g_B(\varphi,s;\xi_1,\tau_1)g_{\widetilde{B}}(A,R;\varphi,s)\,d\varphi\,ds,\end{aligned} \tag{2.4}$$

with C depending on λ, n, T_1, μ, η, the diameter of B and $\min[\operatorname{dist}(\xi_2,\partial B),\operatorname{dist}(\xi_1,\partial B)]$. Using (2.3) and (2.4) in the representation formulas (2.1) and (2.2) we easily conclude the proof.

3. The main result. To state and prove the main result of this paper we need to introduce some notation. Set

$$
\begin{aligned}
B_r(x_0) &= \{ x \in \mathbf{R}^n : |x - x_0| < r \}, \\
C_r(x_0) &= \{ x \in \mathbf{R}^n : |x_j - x_{0j}| < r,\ j = 1, \ldots, n \}, \\
Q_r(x_0, t_0) &= C_r(x_0) \times (t_0 - r^2, t_0), \\
Q_r^+(x_0, t_0) &= C_r(x_0) \times (t_0 + r^2, t_0 + 2r^2), \\
Q_r^\eta(x_0, t_0) &= C_r(x_0) \times (t_0 + 3r^2, t_0 + r^2/\eta), \qquad 0 < \eta < 1/3.
\end{aligned}
$$

A rectangle like $Q_r(x_0, t_0)$ will be called a *parabolic cube*. Also set $g(y, s) \equiv g_B(0, 2; y, s)$, where g_B is the Green's function for L and the cylinder $B_{2n} \times (-\infty, \infty)$. (We choose B_{2n} only to guarantee that $\overline{C_2(0)} \subset B_{2n}$.)

For any measurable set Γ contained in $Q_1 \equiv Q_1(0, 1)$ define

$$w(\Gamma) = \int_\Gamma g(y, s)\, dy\, ds,$$

and denote by $|\Gamma|$ the Lebesgue measure of Γ. In this section we will prove the following result.

THEOREM 5. *There exist two positive constants c and M depending only on λ and n such that $w(\Gamma) \geq c|\Gamma|^M$.*

Proof. We first show that there exists ξ, $0 < \xi < 1$, depending only on λ, n such that if $|\Gamma| \geq \xi$ then $w(\Gamma) \geq c = c(\lambda, n)$ and therefore $w(\Gamma) \geq c|\Gamma|$.

Consider the function

$$w^{x,t}(\Gamma) = \int_\Gamma g_{B_{2n}}(x, t; y, s)\, dy\, ds$$

and write $w^{x,t}(\Gamma) = w^{x,t}(Q_1) - w^{x,t}(Q_1 \setminus \Gamma)$. Hölder's inequality gives

$$w^{x,t}(Q_1 \setminus \Gamma) \leq |Q_1 \setminus \Gamma|^{1/(n+1)} \|g_{B_{2n}}(x, t; \cdot, \cdot)\|_{L^{(n+1)/n}(Q_1)} \cdot$$

From the work of Krylov [6] (see also Tso [11]) $\|g_{B_{2n}}(x, t; \cdot, \cdot)\|_{L^{(n+1)/n}(Q_1)} \leq C(\lambda, n)$ for all (x, t) in $B_{2n} \times (0, \infty)$. Hence

$$w^{x,t}(Q_1 \setminus \Gamma) \leq C(\lambda, n)(1 - \xi)^{1/(n+1)} \tag{3.1}$$

for every $(x, t) \in B_{2n} \times (0, \infty)$. Choose $\psi \in C_0^\infty(Q_1)$, $0 \leq \psi \leq 1$, $\psi \equiv 1$ on $Q_{1/2}(0, 1/2)$. A maximum principle argument gives $w^{x,t}(Q_1) \geq \psi(x, t)/\|L\psi\|_{L^\infty(Q_1)}$ for every $(x, t) \in Q_1$. Therefore $w^{x,t}(Q_1) \geq c(\lambda, n)$ on $Q_{1/2}(0, 1/2)$. From this inequality and (3.1) we conclude that, for ξ close enough to 1, $w^{(x,t)}(\Gamma) \geq c(\lambda, n)$ on $Q_{1/2}(0, 1/2)$.

The maximum principle and Harnack's inequality imply $w(\Gamma) \geq c(\lambda, n) > 0$.

From now on we keep ξ fixed according to the previous argument and we assume $|\Gamma| < \xi$. In this case we can apply a parabolic version of the Calderón–Zygmund decomposition lemma to construct a sequence $\{Q_j\}$ of parabolic subcubes of Q_1, pairwise nonoverlapping, with the properties: $|\Gamma \cap Q_j| > \xi|Q_j|$, $|\Gamma \setminus \bigcup Q_j| = 0$; each Q_j arises as a subdivision into 2^{n+2} congruent (parabolic) subcubes of another parabolic subcube $\widetilde{Q}_j$ with the property $|\Gamma \cap \widetilde{Q}_j| \le \xi|\widetilde{Q}_j|$. We call $\widetilde{Q}_j$ the *antecedent* of Q_j and the union of the $\widetilde{Q}_j$ can be written as a union of nonoverlapping ones. We still denote by $\{\widetilde{Q}_k\}$ the family of these nonoverlapping parabolic cubes and we set $D = \bigcup_k \widetilde{Q}_k$. Now

$$w(\Gamma) = \sum_j \int_{\Gamma \cap Q_j} g(y,s)\,dy\,ds. \tag{3.2}$$

Write $Q_j = Q_{r_j}(x_j, t_j)$, $\widetilde{Q}_j = Q_{2r_j}(\tilde{x}_j, \tilde{t}_j)$, and set $g_j(y,s) = g_{B_{2\eta r_j}(x_j)}(x_j, t_j + 2r_j^2/\eta; y, s)$. For each j, Theorem 4, after translation and dilation, gives

$$\int_{\Gamma \cap Q_j} g(y,s)\,dy\,ds \ge c\,\frac{g(y_j,s_j)}{g_j(y_j,s_j)} \int_{\Gamma \cap Q_j} g_j(y,s)\,dy\,ds,$$

where (y_j, s_j) is any point of Q_j^+ and c depends only on λ and n. Since $|\Gamma \cap Q_j| > \xi|Q_j|$, we can appropriately dilate and use the first part of this proof to obtain

$$\int_{\Gamma \cap Q_j} g_j(y,s)\,dy\,ds \sim \int_{Q_j} g_j(y,s)\,dy\,ds \sim \int_{\widetilde{Q}_j^\eta} g_j(y,s)\,dy\,ds,$$

where the equivalence constants depend only on λ, n and η. Also from Theorem 1, $g(0,2;y_j,s_j) \sim g(0,2+2/\eta;y_j,s_j)$ with the same equivalence dependence. Hence

$$\int_{\Gamma \cap Q_j} g(y,s)\,dy\,ds \ge c\,\frac{g(0,2+2/\eta;y_j,s_j)}{g_j(y_j,s_j)} \int_{\widetilde{Q}_j^\eta} g_j(y,s)\,dy\,ds.$$

Another application of Theorem 3 gives

$$\begin{aligned} w(\Gamma) &\ge c(\lambda,n,\eta) \sum_j \int_{\widetilde{Q}_j^\eta} g(0,2+2/\eta;y,s)\,dy\,ds \\ &\ge c(\lambda,n,\eta) w^\eta\Big(\bigcup_j \widetilde{Q}_j^\eta\Big) = c(\lambda,n,\eta) w^\eta(D^\eta), \end{aligned} \tag{3.3}$$

where $w^\eta(E) \equiv \int_E g(0,2+2/\eta;y,s)\,dy\,ds$, $D^\eta = \bigcup_j \widetilde{Q}_j^\eta$.

We now distinguish two cases that we treat by means of the following lemmas:

LEMMA 6. *Suppose that for some fixed positive δ we have*

$$|D^\eta \setminus Q_1| \ge \delta|\Gamma|. \tag{3.4}$$

Then there exists a $\widetilde{Q}_k^\eta$ *such that*

$$|\widetilde{Q}_k^\eta| \geq c(n, \eta, \delta)|\Gamma|^{(n+1)/2}. \tag{3.5}$$

Proof. Let r_k be the side length of $\widetilde{Q}_k^\eta$. Either there exists k such that $r_k^2 \geq (\delta/6)|\Gamma|$ (and in this case (3.5) holds) or for each k the opposite inequality holds. If this is the case, observe that there must be a $\widetilde{Q}_k^\eta$ whose side length is greater than $(\delta/2)|\Gamma|$. In fact, the bottom level of each $\widetilde{Q}_j^\eta$ is less than $1 + (\delta/2)|\Gamma|$; if each $\widetilde{Q}_j^\eta$ had top level less than $1 + \delta|\Gamma|$ we would have

$$D^\eta \subset Q_1 \cup (B_1 \times (1, 1 + \delta|\Gamma|))$$

and therefore $|D^\eta \setminus Q_1| < \delta|\Gamma|$, which contradicts (3.4). We conclude that there exists $c = c(\eta, \delta)$ and a $\widetilde{Q}_k^\eta$ whose side length r_k satisfies the condition $r_k \geq c(\eta, \delta)|\Gamma|^{1/2}$. This shows (3.5).

LEMMA 7. *There exist positive numbers c, δ, η, ρ with $\eta < 1/3$, $\rho > 1$, depending only on λ, n, such that if $|D^\eta \setminus Q_1| < \delta|\Gamma|$ then $w(\Gamma) \geq cw(\Gamma_1)$ where $\Gamma_1 \subset Q_1$ and $|\Gamma_1| \geq \rho|\Gamma|$. (In fact $\Gamma_1 = D^\eta \cap Q_1$.)*

Proof. From [8, Lemma 2.2, p. 157] we have

$$|D^\eta| \geq \frac{1 - 3\eta}{1 + \eta}|D|.$$

Set $\Gamma_1 = D^\eta \cap Q_1$. Then

$$|\Gamma_1| \geq \frac{1 - 3\eta}{1 + \eta}|D| - \delta|\Gamma|,$$

and since $|\Gamma| \leq \sum_k |\Gamma \cap \widetilde{Q}_k| \leq \xi \sum_k |\widetilde{Q}_k| = \xi|D|$,

$$|\Gamma_1| \geq \left[\frac{1 - 3\eta}{(1 + \eta)\xi} - \delta\right]|\Gamma|.$$

Now set

$$\rho = \frac{1 - 3\eta}{(1 + \eta)\xi} - \delta.$$

Choose δ, η small enough to have $\rho > 1$ and then use Harnack's inequality [7] in (3.3).

We now conclude the proof of Theorem 4. Choose η and δ according to Lemma 7. If (3.4) holds, select a $\widetilde{Q}_k^\eta$ for which (3.5) occurs. From Theorem 1

$$\int_{\widetilde{Q}_k^\eta} g(0, 2 + 2/\eta; y, s)\, dy\, ds \geq c \int_{\widetilde{Q}_k^\eta} g(\tilde{x}_k, 2 + 2/\eta; y, s)\, dy\, ds.$$

We clearly decrease the last integral by replacing g by $g_{B_{2nr_k}(\bar{x}_k)}$. We then translate and dilate appropriately and apply Harnack's inequality to obtain

$$\int_{\widetilde{Q}_k^\eta} g_{B_{2nr_k}(\bar{x}_k)}(x_k, 2+2/\eta; y, s)\, dy\, ds \geq c|\Gamma|^M$$

where c and M depend only on n and η.

Assume now $|D^\eta \setminus Q_1| < \delta|\Gamma|$. By Lemma 6 we have either

$$w(\Gamma) \geq c(\lambda, n)|\Gamma|^M = c|\Gamma|^M$$

or

$$w(\Gamma) \geq c(\lambda, n) w(\Gamma_1) = c w(\Gamma_1),$$

with $\Gamma_1 \subset Q_1$ and $|\Gamma_1| \geq \rho|\Gamma|$. In the second case repeat for Γ_1 the entire procedure done for Γ. Again we find either $w(\Gamma_1) \geq c|\Gamma_1|^M$, and, therefore, $w(\Gamma) \geq c^2\rho^M|\Gamma|^M$, or $w(\Gamma_1) \geq c w(\Gamma_2)$ where $\Gamma_2 \subset Q_1$ and $|\Gamma_2| \geq \rho|\Gamma_1| \geq \rho^2|\Gamma|$. Proceeding in this way we construct a sequence of sets $\{\Gamma_k\}$, stopping the process either when

$$w(\Gamma_k) \geq c|\Gamma_k|^M$$

and therefore $w(\Gamma) \geq c^{k+1}\rho^{kM}|\Gamma|^M$, or when

$$\rho^{k-1} \leq \xi/|\Gamma| < \rho^k.$$

In this last case we have $|\Gamma_k| \geq \rho^k|\Gamma| > \xi$ and hence, $w(\Gamma_k) \geq \bar{c}(\lambda, n)$. Since $w(\Gamma) \geq c^k w(\Gamma_k)$ and $k \sim (1/\log\delta)\log(\xi/|\Gamma|)$, we conclude that $w(\Gamma) \geq c(\lambda, n)|\Gamma|^M$ with $M = M(\lambda, n)$.

A simple consequence of Theorem 5 is the following integrability property of the Green's function g.

COROLLARY 8. *There exist two positive constants ε_0 and C depending only on n, λ such that*

$$\int_{Q_1} g^{-\varepsilon_0}(y, s)\, dy\, ds \leq C. \tag{3.6}$$

Proof. Let $\Gamma_t \equiv \{(y, s) \in Q_1;\ g(y, s) < 1/t\}$; then Theorem 5 gives

$$c|\Gamma_t|^M \leq \int_{\Gamma_t} g(y, s)\, dy\, ds < \frac{1}{t}|\Gamma_t|$$

or $|\Gamma_t| \leq c(n, \lambda)t^{-1/(M-1)}$. We conclude that (3.6) holds with $\varepsilon_0 = 1/M$.

4. Estimates on the first and second derivatives of solutions of $Lu = f$. In this section, following the technique of L. C. Evans [2] and F. Lin [9], we give a priori estimates in terms of Lu of a small positive power of first and second spatial derivatives of functions u which vanish on the parabolic boundary of a cylinder. These results are achieved by first establishing an

L^2-integrability of a first order derivative and an L^1-integrability of a second order derivative, each with respect to a measure $g(y,s)\,dy\,ds$ where g is a Green's function associated with a parabolic operator.

Let us begin with the spatial gradient estimate.

Set $D_1 = B_1 \times (0,1)$ and $g(y,s) = g_{B_2}(0,2;y,s)$, the Green's function corresponding to B_2 and the operator L with smooth coefficients.

THEOREM 9. *Assume $u = u(x,t)$ is a smooth function in D_1 vanishing on the parabolic boundary $\partial_p D_1$. Then there exists a positive constant C depending only on λ and n such that*

$$\int_{D_1} |\nabla_x u|^2 g \leq C\|Lu\|^2_{L^{n+1}(D_1)} .$$

Proof.

$$\begin{aligned}\int_{D_1} |\nabla_x u|^2 g &\leq \frac{1}{\lambda} \int_{D_1} \sum_{i,j=1}^{n} a_{ij}(x,t)(D_{x_i}u)(D_{x_j}u)g\,dx\,dt \\ &= \frac{1}{2\lambda}\int_{D_1} L(u^2)g - \frac{1}{\lambda}\int_{D_1} u(Lu)g.\end{aligned}$$

Since $u = 0$ on $\partial_p D_1$ and $g > 0$ inside D_1, an integration by parts yields $\int_{D_1} L(u^2)g \leq 0$. Therefore

$$\int_{D_1} |\nabla_x u|^2 g \leq \frac{1}{\lambda}\|u\|_{L^\infty(D_1)}\|g\|_{L^{(n+1)/n}(D_1)}\|Lu\|_{L^{n+1}(D_1)} .$$

The conclusion follows easily from the result of Krylov [6]. (See also [11].)

COROLLARY 10. *There exist two positive constants C and σ depending only on λ, n such that for any smooth u vanishing on $\partial_p D_1$,*

$$\int_{D_1} |\nabla_x u|^\sigma \leq C\|Lu\|^\sigma_{L^{n+1}(D_1)} . \tag{4.1}$$

Proof. By Corollary 8 there exists ε such that $\int_{D_1} g^{-\varepsilon} < c(\lambda,n)$. Then (4.1) follows from Theorem 9 and Hölder's inequality with $\sigma = 2\varepsilon/(1+\varepsilon)$.

We now consider second derivatives. Set $D_{1/2} = B_{1/2} \times (0,1/2)$.

THEOREM 11. *There exist positive constants C and σ depending only on λ, n such that for every smooth function u vanishing on $\partial_p D_1$ we have for $i,j = 1,\ldots,n$*

$$\int_{D_{1/2}} |D^2_{x_i x_j} u|^\sigma \leq C\|Lu\|^\sigma_{L^{n+1}(D_1)} . \tag{4.2}$$

Proof. Using an observation of Fang-Hua Lin [9, p. 449, (2.3)] given u and L we can construct an operator L_u with coefficients depending on u such that

$$L_u u(x,t) = Lu(x,t) + \gamma(x,t)\|D_x^2 u(x,t)\| \tag{4.3}$$

where $0 < C(\lambda)^{-1} < \gamma(x,t) < C(\lambda) < \infty$ and

$$\|D_x^2 u(x,t)\| = \Big[\sum_{i,j=1}^{n} |D_{x_i x_j}^2 u(x,t)|^2\Big]^{1/2}.$$

Moreover, the parabolicity constant of L_u depends only on that of L.

We now construct a Green's function for L_u in D_1 with pole at $(0,1)$. More precisely, we construct a function $g_u(y,s) = g_{L_u,B_1}(0,1;y,s)$ with the following properties:

(i) $0 \le g_{L_u,B_1}(0,1;y,s) \in L^{(n+1)/n}(D_1)$ and

$$\int_{D_1} \{g_{L_u,B_1}(0,1;y,s)\}^{(n+1)/n}\, dy\, ds \le C(\lambda,n).$$

(ii) There exists $\varepsilon = \varepsilon(\lambda,n)$ such that

$$\int_{D_{1/2}} g_{L_u,B_1}^{-\varepsilon}(0,1;y,s)\, dy\, ds \le C(\lambda,n).$$

(iii) If v is a smooth function vanishing on $\partial_p D_1$ then

$$v(0,1) = -\int_{D_1} g_{L_u,B_1}(0,1;y,s) L_u v(y,s)\, dy\, ds.$$

To construct such a function we first regularize the coefficients of L_u and denote by L_u^k the resulting operator. Let $g_k(y,s) = g_{L_u^k,B_1}(0,1;y,s)$. Again we have

$$\|g_k\|_{L^{(n+1)/n}(D_1)} \le C(\lambda,n), \tag{4.4}$$

so that a subsequence converges weakly in $L^{(n+1)/n}(D_1)$ to a function g_u which satisfies (4.4). Clearly (ii) holds for g_u since Theorem 5 holds for g_k with constants depending only on λ and n. Also property (iii) is a consequence of the fact that it holds for every g_k. Then, from (4.3) we have

$$\begin{aligned} -u(0,1) = &\int_{D_1} g_u(y,s) L_u u(y,s)\, dy\, ds \\ &+ \int_{D_1} g_u(y,s)\gamma(y,s)\|D_x^2 u(y,s)\|\, dy\, ds. \end{aligned}$$

Using once more [6] or [11], property (i) of g_u, and the lower bound for γ, we conclude

$$\int\limits_{D_1} g_u(y,s)\|D_x^2u(y,s)\|\,dy\,ds \le c\|Lu\|_{L^{n+1}(D_1)}\,.$$

The same argument as in the proof of Corollary 10 gives (4.2) with $\sigma = \varepsilon/(1+\varepsilon)$.

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