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THE LOCAL WEIGHT OF AN EFFECTIVE LOCALLY COMPACT TRANSFORMATION GROUP AND THE DIMENSION OF $L^2(G)$

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1. Introduction. Throughout this note, G denotes a topological group with identity element e. All topological spaces are to satisfy the Hausdorff separation property.

A *transformation group* is a triple (G, X, π), where X is a topological space and $\pi\colon G\times X\to X$ is a separately continuous function satisfying the following conditions:

(i) $\forall x\in X\ [\pi(e, x) = x]$;

(ii) $\forall s, t\in G,\ \forall x\in X\ [\pi(s, \pi(t, x)) = \pi(st, x)]$.

If (G, X, π) is a transformation group, then a function $\pi^t\colon X\to X$ may be defined by $\pi^t(x) = \pi(t, x)$ $(x\in X)$. Observe that $t\mapsto \pi^t$ is a homomorphism of G into the group of all autohomeomorphisms of X. The transformation group (G, X, π) is said to be *effective* whenever this homomorphism is an isomorphism, i.e. whenever

$$\forall t\in G\ [t\neq e \Rightarrow \exists x\in X\ [\pi(t, x)\neq x]].$$

The transformation group (G, X, π) is said to be *free* whenever

$$\forall t\in G\ [t\neq e \Rightarrow \forall x\in X\ [\pi(t, x)\neq x]].$$

In this note we consider inequalities for the local weight of G if G is locally compact. Recall that for any topological space X the following cardinal numbers are unambiguously defined (for any set A, $|A|$ denotes the cardinality of A):

the *local weight of X at $x\in X$*:

$$\chi(x, X) := \min\{|\mathscr{V}|\colon \mathscr{V} \text{ is a local base at } x\};$$

the *local weight* of X:

$$\chi(X) := \sup\{\chi(x, X)\colon x\in X\}.$$

The *weight* of X is defined by

$$w(X) := \min\{|\mathscr{U}|: \mathscr{U} \text{ is an open base for } X\},$$

and the *density* of X is determined by

$$d(X) := \min\{|A|: A \text{ is dense in } X\}.$$

The inequality which we shall consider concerns $\chi(G)$ on the one hand and $\chi(X)$ and the cardinal numbers of certain subsets of X on the other hand, where (G, X, π) is a transformation group. To specify these subsets of X we give the following definition:

Definition. An *action-determining set* in a transformation group (G, X, π) is a subset A of X such that

$$\forall t \in G \; [\pi^t|_A = \pi^e|_A \Rightarrow t = e].$$

Examples. 1. A transformation group (G, X, π) is free iff $\{x\}$ is an action-determining subset of X for each $x \in X$.

2. A transformation group (G, X, π) is effective iff X is an action-determining set, iff each dense subset of X is action-determining, iff some dense subset of X is action-determining.

3. Let G be locally compact and consider for $p \geqslant 1$ the transformation group $(G, L^p(G), \tilde{\varrho})$, where $\tilde{\varrho}$ denotes right translation, i.e. $\tilde{\varrho}(t, f)(s) = f(st)$ for all $f \in L^p(G)$ and $s, t \in G$. Here the space $L^p(G)$ is with respect to right Haar measure on G, so that, indeed, $\tilde{\varrho}$ is separately continuous (cf. [2], 20.1 and 20.4). It is easy to see that this transformation group is effective. Indeed, for any $t \in G$, $t \neq e$, there is a continuous function f on G with compact support such that $\tilde{\varrho}^t f(e) \neq f(e)$, hence $\tilde{\varrho}^t f \neq f$ in $L^p(G)$. In particular, $(G, L^2(G), \tilde{\varrho})$ is an effective transformation group, so that each dense subset of $L^2(G)$ is action-determining. Hence $L^2(G)$ contains action-determining subsets of cardinality $d(L^2(G))$. However, $L^2(G)$ is a Hilbert space, and if $\delta(L^2(G))$ denotes the cardinality of an orthonormal base for $L^2(G)$, then we have

$$\aleph_0 \cdot \delta(L^2(G)) = d(L^2(G)) = w(L^2(G)).$$

Hence $L^2(G)$ contains action-determining subsets of cardinality $\delta(L^2(G))$, the (Hilbert) dimension of $L^2(G)$, provided $\delta(L^2(G)) \geqslant \aleph_0$ [1].

4. $(G, L^1(G), \tilde{\varrho})$ is an effective transformation group as well. It is known that $L^1(G)$ contains an *approximate unit* with respect to convolution, i.e. a net $\{f_\lambda\}_{\lambda \in \Lambda}$ in $L^1(G)$ such that

$$\lim_{\lambda \in \Lambda} f * f_\lambda = f \quad \text{for every } f \in L^1(G)$$

(1) Using the fact that each ϱ^t is linear, it is easily seen that this is also true if $\delta(L^2(G))$ is finite.

(cf. [2], 20.27). We claim that *any approximate unit for $L^1(G)$ is an action-determining set in $(G, L^1(G), \tilde{\varrho})$*. To prove this claim, suppose that $\{f_\lambda\}_{\lambda\in\Lambda}$ is an approximate unit in $L^1(G)$ and that $t \in G$ satisfies $\tilde{\varrho}^t f_\lambda = f_\lambda$ for every $\lambda \in \Lambda$. Then for all $f \in L^1(G)$ we have

$$\tilde{\varrho}^t f = \lim_{\lambda\in\Lambda} \tilde{\varrho}^t(f*f_\lambda) = \lim_{\lambda\in\Lambda} f*\tilde{\varrho}^t f_\lambda = \lim_{\lambda\in\Lambda} f*f_\lambda = f,$$

hence $t = e$, by effectiveness.

2. Basic result. Our starting point is the following well-known lemma:

LEMMA 1. *Let Y be a topological space, let $x_0 \in Y$ and suppose that $\mathscr{B}$ is a set of neighbourhoods of x_0 such that $\bigcap \mathscr{B} = \{x_0\}$. Then, if x_0 has a compact neighbourhood, $\chi(x_0, Y) \leqslant |\mathscr{B}|$.*

Proof. If $\{x_0\}$ is isolated, then $\chi(x_0, Y) = 1$, and the lemma is trivial. So we may suppose that $|\mathscr{B}| \geqslant \aleph_0$. Let $\mathscr{B}^*$ denote the family of all finite intersections of members of $\mathscr{B}$. Then $|\mathscr{B}^*| = |\mathscr{B}|$. By [1], Corollary 2 to Theorem 3.1.3, $\mathscr{B}^*$ is a base at x_0, hence $\chi(x_0, Y) \leqslant |\mathscr{B}^*| = |\mathscr{B}|$.

THEOREM 1. *Let (G, X, ϱ) denote a transformation group and let A be an action-determining set in X. If G is locally compact, then*

$$\chi(G) \leqslant |A| \sup_{a\in A} \chi(a, X) \leqslant |A| \chi(X).$$

Proof. For $a \in A$, let $\mathscr{B}_a$ denote a local base at a such that $|\mathscr{B}_a| = \chi(a, X)$. For any $a \in A$ and $V \in \mathscr{B}_a$, the set

$$\mathscr{U}(a, V) := \{t \in G:\ \pi(t, a) \in V\}$$

is a neighbourhood of e in G. Since A is action-determining, we have

$$\bigcap_{a\in A} \bigcap_{V\in\mathscr{B}_a} \mathscr{U}(a, V) = \{e\}$$

(here we use only the T_1-separation property for X). From Lemma 1 it follows that

$$\chi(G) \leqslant |A| \sup_{a\in A} |\mathscr{B}_a| = |A| \sup_{a\in A} \chi(a, X).$$

COROLLARY 1. *Let G be a locally compact topological group and (G, X, π) an effective transformation group. Then $\chi(G) \leqslant d(X)\chi(X)$. If (G, X, π) is free, then*

$$\chi(G) \leqslant \min\{\chi(x, X):\ x \in X\}.$$

In particular, if G acts effectively [freely] on a separable first countable space [on a first countable space], then G is metrizable.

For the proof apply Theorem 1 and [2], 8.3.

Remark. The result that G is metrizable whenever G acts effectively on a separable first countable space has been known for a long time (cf. [3], 2.11).

COROLLARY 2. *Let G be a locally compact group and let $\{f_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit of $L^1(G)$. Then $\chi(G) \leqslant |\Lambda|$.*

Proof. If $|\Lambda| < \aleph_0$, then $L^1(G)$ has a unit, hence G is discrete, and $\chi(G) = 1 \leqslant |\Lambda|$. In the other case, it follows from Corollary 1 and Example 4 in Section 1 that $\chi(G) \leqslant |\Lambda| \cdot \aleph_0 = |\Lambda|$.

Remark. By [2], 20.27, $L^1(G)$ has an approximate unit of cardinality $\chi(G)$ (if $\chi(G) < \aleph_0$, then G is discrete, and $L^1(G)$ has a unit, i.e. an approximate unit of cardinality $1 = \chi(G)$). Hence the least cardinal number of a directed set defining an approximate unit for $L^1(G)$ equals $\chi(G)$. A proof for this fact is also indicated in [2], 28.70(b), but our proof seems to be simpler.

3. The weight of $L^2(G)$. Let G denote a locally compact topological group. It is well known that the dimension $\delta(L^2(G))$ of the Hilbert space $L^2(G)$ satisfies the inequality $\delta(L^2(G)) \leqslant w(G)$ (cf. [2], the proof of Theorem 24.14). For compact groups, it is known that $\delta(L^2(G)) = w(G)$ (cf. [2], 28.2) and we shall show now that this equality holds for arbitrary locally compact groups. In our proof we shall use the concept of the *Lindelöf degree* of a topological space X, i.e. the cardinal number

$$\mathscr{L}(X) := \min\{\aleph\colon \text{each open covering of } X \text{ has a subcovering of cardinality } \aleph\}.$$

Since G is locally compact, $\mathscr{L}(G) \leqslant \aleph_0$ iff G is σ-compact.

LEMMA 2. $w(G) \leqslant \chi(G)\mathscr{L}(G)$.

Proof. Let $\mathscr{V}$ denote a local base at e with $|\mathscr{V}| = \chi(G)$. For each $U \in \mathscr{V}$, let $\mathscr{F}_U$ be a covering of G by $\mathscr{L}(G)$ left translates of U. Then $\bigcup\{\mathscr{F}_U\colon U \in \mathscr{V}\}$ is a base for the topology of G of cardinality $\chi(G)\mathscr{L}(G)$.

LEMMA 3. $\mathscr{L}(G) \leqslant \delta(L^2(G))$.

Proof. For finite groups we have $\mathscr{L}(G) = |G| = \delta(L^2(G))$, so we may assume that G is infinite. Then there is a family $\mathscr{W}$ of pairwise disjoint, non-empty open subsets of G such that $|\mathscr{W}| \geqslant \mathscr{L}(G)$. Indeed, if G is σ-compact, let

$$\mathscr{W} = \{U_n \sim \overline{U}_{n+1}\colon n \in N\}$$

for some suitable sequence $\{U_n\colon n \in N\}$ of neighbourhoods of e, and if G is not σ-compact, take for $\mathscr{W}$ the family of all left cosets of an open σ-compact subgroup of G (such a subgroup exists by [2], 5.7, with compact U).

For each $W \in \mathscr{W}$, let f_W be a continuous function with compact support contained in W, $0 \neq f_W \geqslant 0$. After suitable normalization, $\{f_W\colon W \in \mathscr{W}\}$ is an orthonormal subset of $L^2(G)$ of cardinality $|\mathscr{W}|$. Hence

$$\delta(L^2(G)) \geqslant |\mathscr{W}| \geqslant \mathscr{L}(G).$$

Remark. It is clear that $\mathscr{L}(G) < \aleph_0$ iff G is finite. So it follows from Lemma 3 that G is finite whenever $\delta\big(L^2(G)\big)$ is finite (cf. [2], 28.1; compare also the proof given there with our proof of Lemma 3).

LEMMA 4. $\chi(G) \leqslant \delta\big(L^2(G)\big)$.

Proof. If $\delta\big(L^2(G)\big) < \aleph_0$, then G is finite, hence $\chi(G) = 1 \leqslant \delta\big(L^2(G)\big)$. If $\delta\big(L^2(G)\big) \geqslant \aleph_0$, then the result follows from Theorem 1 and Example 3 in Section 1.

THEOREM 2. *For any locally compact topological group G we have* $w(G) = \delta\big(L^2(G)\big)$.

Proof. The inequality $\geqslant$ was known before, and $\leqslant$ follows from Lemmas 2-4.

Added in proof. The following observation was pointed out to the author by Prof. M. Wilhelm:

An immediate consequence of Theorem 2 is that $w\,\big(L^2(G)\big) = w(G)$ for any locally compact group G. As for such groups G all L^p-spaces are homeomorphic by a result of S. Mazur (cf. Bourbaki, *Integration*, IV, § 6.5, Exercise 10), we obtain $w\big(L^p(G)\big) = w(G)$ for $1 \leqslant p \leqslant \infty$.

REFERENCES

[1] R. Engelking, *Outline of general topology*, Amsterdam 1968.

[2] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Vol. I and II, Berlin - Heidelberg - New York 1963 and 1970.

[3] D. Montgomery and L. Zippin, *Topological transformation groups*, New York 1955.

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