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EXTENDING METRICS UNIFORMLY

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0. Introduction. The problem of extending metrics or complete metrics has been investigated by several authors, e.g., Hausdorff [11], [12], Bing [8], Arens [1], Bacon [5], Toruńczyk [18]. The following theorem is known:

THEOREM 0. *Let A be a closed subset of a metric space X with a metric d. If ϱ is any metric on A equivalent to d on A, then ϱ can be extended to a metric $\bar{\varrho}$ on X which is equivalent to d.*

Moreover, if d and ϱ are complete metrics, then $\bar{\varrho}$ can be taken to be complete.

In this note we consider the question under what assumptions the metric ϱ can be extended to a metric on X which is uniformly equivalent to d (i.e., induces the same uniformity as d). The following example shows that such an extension need not exist, even if ϱ is uniformly equivalent to d on A.

Example. Let R denote the real line, N — natural numbers, and let the metric ϱ on N be defined by

$$\varrho(m, n) = |m^2 - n^2|.$$

It is clear that ϱ is uniformly equivalent to the metric $|\cdot|$ on R. If $\bar{\varrho}$ were any metric on R which extends ϱ, then, for each $n \in N$, since $\bar{\varrho}(n, n+1) = 2n+1$, there would exist points $x_n, y_n \in [n, n+1]$ such that $|x_n - y_n| \leqslant 1/n$ and $\bar{\varrho}(x_n, y_n) \geqslant 1$. This shows that no extension of ϱ is uniformly equivalent to $|\cdot|$.

In this note we prove that under some additional assumptions the metric ϱ can be extended over X. In Section 1 we give a short proof of Theorem 0. The ideas of that proof will be used further on. In Section 2, which contains the main result of this note, we prove that if A is a subset of a metric space (X, d) and ϱ is a metric on A uniformly equivalent to $d \mid A \times A$, then ϱ can be extended to a metric $\bar{\varrho}$ on X uniformly equivalent to d if and only if there is a uniformly continuous pseudometric $\tilde{\varrho}$ on

(X, d) such that $\tilde{\varrho}|A \times A = \varrho$. Thus the problem of extending ϱ uniformly is related to that of extending a certain uniformly continuous map. Using this, we apply in Sections 2 and 3 the results of Aronszajn and Panitchpakdi [2], Katětov [15], Ramer [17], and Atsuji [3], [4] to get some sufficient conditions for the metric ϱ to have the required extension (e.g., if ϱ is bounded, then that extension does exist).

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1. Proof of Theorem 0. For a set D, let $m(D)$ denote the space of all bounded real-valued functions on D (with the supremum norm).

In the notation of Theorem 0 we define a map $j\colon X \to m(X)$ by

$$(jx)y = d(x, y) - d(x_0, y) \quad \text{for } x, y \in X,$$

where x_0 is any fixed point in A. Let further $f\colon A \to m(A)$ be defined by

$$(fx)y = \varrho(x, y) - \varrho(x_0, y) \quad \text{for } x, y \in A.$$

It is well known [7] that $j\colon (X, d) \to m(X)$ and $f\colon (A, \varrho) \to m(A)$ are isometric embeddings.

By a Dugundji theorem [10] there is a continuous map $\tilde{f}\colon X \to m(A)$ with $\tilde{f}|A = f$. Define now a map $g\colon X \to m(A) \times R$ by

$$g(x) = \big(\tilde{f}(x), d(x, A)\big).$$

It is clear that $g(A)$ is closed in $g(X)$ and $g|A$ is an isometric embedding of (A, ϱ) into $m(A) \times R$.

Using the Dugundji theorem again, we take a map $u\colon g(X) \to m(X)$ with $ug(x) = j(x)$ for every $x \in A$. We now define $\bar{\varrho}$ by

$$(1) \quad \bar{\varrho}(x, y) = \|g(x) - g(y)\|_1 + \|j(x) - j(y) - ug(x) + ug(y)\|_2 \quad \text{for } x, y \in X,$$

where $\|\cdot\|_1$ is the norm of $m(A) \times R$ and $\|\cdot\|_2$ is the norm of $m(X)$. It is easy to check that $\bar{\varrho}$ extends ϱ and is equivalent to d.

If, moreover, (X, d) and (A, ϱ) are complete metric spaces, then $g(A)$ is closed in $m(A) \times R$ and, therefore, u can be taken as a map from $m(A) \times R$ into $m(X)$.

Let $\{x_n\}$ be a $\bar{\varrho}$-Cauchy sequence of points in X. Then, by (1), $\{g(x_n)\}$ is a Cauchy sequence of points in $m(A) \times R$. Hence $\{g(x_n)\}$ converges to a point of $m(A) \times R$. Since u is continuous, we infer that $\{ug(x_n)\}$ is a Cauchy sequence in $m(X)$. Thus by (1) we get

$$\lim_{n,m\to\infty} d(x_n, x_m) = \lim_{n,m\to\infty} \|j(x_n) - j(x_m)\| = 0.$$

Since (X, d) is complete, there exists a point $x_0 \in X$ such that $\lim d(x_n, x_0) = 0$. And since $\bar{\varrho}$ is equivalent to d on X, we get $\bar{\varrho}(x_n, x_0) \to 0$; this shows the completeness of $\bar{\varrho}$ and completes the proof of the theorem.

2. Uniform extension of metrics. In the sequel we need the following lemma due to Banach [6], Aronszajn and Panitchpakdi [2], Cipszer and Geher [9]:

2.1. LEMMA. *Let (X, d) be a metric space, let A be a subset of X, and let f be a Lipschitz map from A into $m(D)$ for some set D. Then there exists an extension $\tilde{f}\colon X \to m(D)$ of f which satisfies the Lipschitz condition with the same Lipschitz constant as f.*

2.2. THEOREM. *Let A be a subset of a metric space (X, d) and let ϱ be a metric on A uniformly equivalent to d on A. Then the following conditions are equivalent:*

(a) *There exists a uniformly continuous pseudometric $\tilde{\varrho}$ on X such that $\tilde{\varrho}(x, y) = \varrho(x, y)$ for every $x, y \in A$.*

(b) *There exists a metric $\bar{\varrho}$ on X uniformly equivalent to d on X and such that $\bar{\varrho}(x, y) = \varrho(x, y)$ for every $x, y \in A$.*

Proof. Of course, we have only to prove that (a) implies (b). Let $\tilde{\varrho}$ denote a uniformly continuous pseudometric on X which extends ϱ.

Define $g\colon X \to m(X)$ by $(gx)y = \min\{1, d(x, y)\}$ for $x, y \in X$, and let $g_0 = g|A$. Then $g_0\colon (A, \varrho) \to m(X)$ is bounded and uniformly continuous. By a theorem of Katětov [15] there exists a uniformly continuous map $\tilde{g}_0\colon X/\tilde{\varrho} \to m(X)$ such that, denoting by $\pi\colon X \to X/\tilde{\varrho}$ the natural projection, we have $g_0\pi|A = g|A$ and

$$\sup_{x\in X} \|\tilde{g}_0\pi(x)\| = \sup_{x\in A} \|g_0\pi(x)\|.$$

We now define $\bar{\varrho}$ by

$$\bar{\varrho}(x, y) = \max\{\tilde{\varrho}(x, y), \|g(x)-g(y)-\tilde{g}_0\pi(x)+\tilde{g}_0\pi(y)\|\}. \tag{2}$$

It is easy to see that $\bar{\varrho}$ is a metric on X which extends ϱ; let us show that $\bar{\varrho}$ and d are uniformly equivalent.

Assume that $d(x_n, y_n) \to 0$. Since $\tilde{\varrho}$ is uniformly continuous with respect to d and $g_0\colon (X/\tilde{\varrho}, \tilde{\varrho}) \to m(X)$ is uniformly continuous, we get $\|\tilde{g}_0\pi(x_n)-\tilde{g}_0\pi(y_n)\| \to 0$. And since $\|g(x_n)-g(y_n)\| \to 0$, we infer by (2) that $\bar{\varrho}(x_n, y_n) \to 0$.

Conversely, let $\bar{\varrho}(x_n, y_n) \to 0$. By (2), $\tilde{\varrho}(x_n, y_n) \to 0$. By the uniform continuity of $\tilde{g}_0\colon (X/\tilde{\varrho}, \tilde{\varrho}) \to m(X)$ we have

$$\|\tilde{g}_0\pi(x_n)-\tilde{g}_0\pi(y_n)\| \to 0.$$

Therefore, by (2), $\|g(x_n)-g(y_n)\| \to 0$. Thus $d(x_n, y_n) \to 0$. This shows that d and $\bar{\varrho}$ are uniformly equivalent. The proof is completed.

Remark. In the notation of Theorem 2.2, there are increasing subadditive functions $a_1, a_2\colon [0, \infty) \to [0, \infty)$ such that $a_1 \leqslant a_2$, $\lim\limits_{t\to 0} a_i(t)$

$= 0 = a_i(0)$ for $i = 1, 2$ and

(3) $$a_1 d(x, y) \leqslant \varrho(x, y) \leqslant a_2 d(x, y) \quad \text{for } x, y \in A,$$

then by Lemma 2.1 there exists a pseudometric $\tilde{\varrho}$ on X such that

$$\tilde{\varrho}(x, y) \leqslant a_2 d(x, y) \quad \text{for } x, y \in X.$$

Hence the formula

$$\bar{\varrho}(x, y) = \max\{\tilde{\varrho}(x, y),\ a_1 d(x, y)\} \quad \text{for } x, y \in X$$

defines a metric $\bar{\varrho}$ on X which extends ϱ and satisfies (3), with ϱ replaced by $\bar{\varrho}$ for all $x, y \in X$. In particular, if ϱ is Lipschitz equivalent to $d|A \times A$, then ϱ can be extended to a metric $\bar{\varrho}$ on X which is Lipschitz equivalent to d. However, if ϱ is merely assumed to be uniformly equivalent to $d|A \times A$, then the functions a_1, a_2 satisfying (3) need not exist.

2.3. COROLLARY. *Let A be a subset of a metric space (X, d). Then the following conditions are equivalent:*

(a) *Any uniformly continuous map $f: (A, d) \to m(A)$ has a uniformly continuous extension $\tilde{f}: (X, d) \to m(A)$.*

(b) *Any metric ϱ on A uniformly equivalent to $d|A \times A$ can be extended to a metric $\bar{\varrho}$ on X uniformly equivalent to d on X.*

Proof. (a) $\Rightarrow$ (b). Let ϱ be a metric uniformly equivalent to d on A. The map $f: (A, d) \to m(A)$, defined by

$$(fx)y = \varrho(x, y) - \varrho(x_0, y) \quad \text{for } x, y \in A,$$

is uniformly continuous. By (a), there exists a uniformly continuous map $\tilde{f}: (X, d) \to m(A)$ such that $\tilde{f}|A = f$. Putting

$$\tilde{\varrho}(x, y) = \|\tilde{f}(x) - \tilde{f}(y)\|,$$

we get a uniformly continuous pseudometric $\tilde{\varrho}$ on X which extends ϱ. From 2.2 it now follows that there exists a metric $\bar{\varrho}$ on X which is uniformly equivalent to d and extends ϱ.

(b) $\Rightarrow$ (a). Let $f: A \to m(A)$ be a uniformly continuous map. Putting

$$\varrho(x, y) = \|f(x) - f(y)\| + d(x, y) \quad \text{for } x, y \in A,$$

we get a metric ϱ on A which is uniformly equivalent to $d|A \times A$. By (b), there is a metric $\bar{\varrho}$ which is uniformly equivalent to d on X and extends ϱ. Since $f: (A, \bar{\varrho}) \to m(A)$ satisfies the Lipschitz condition, there exists, by 2.1, a Lipschitz map $\tilde{f}: (X, \bar{\varrho}) \to m(A)$ such that $\tilde{f}|A = f$ and $\|\tilde{f}(x) - \tilde{f}(y)\| \leqslant \bar{\varrho}(x, y)$ for all $x, y \in X$. It can easily be seen that $\tilde{f}$ is uniformly continuous with respect to the metric d, as required.

2.4. PROPOSITION. *Let A be a subset of a metric space (X, d) and let ϱ be a metric uniformly equivalent to d on A. Assume further that*

$$\limsup_{t\to\infty} t^{-1}\omega_{\varrho/d}(t) < \infty, \tag{4}$$

where

$$\omega_{\varrho/d}(t) = \sup\{\varrho(x, y), x, y \in A: d(x, y) \leqslant t\}. \tag{5}$$

Then there exists a metric $\bar{\varrho}$ on X which is uniformly equivalent to d and extends ϱ.

Proof. By a lemma of Aronszajn and Panitchpakdi [2], condition (4) implies that there is a subadditive function φ: $[0, \infty) \to [0, \infty)$ such that

$$\lim_{t\to 0}\varphi(t) = 0 = \varphi(0),$$

$$\varphi(t) > 0 \quad \text{for } t > 0,$$

and

$$\varphi(t) \geqslant \omega_{\varrho/d}(t) \quad \text{for every } t \in [0, \infty).$$

Let f: $A \to m(A)$ be a map defined by

$$(fx)y = \varrho(x, y) - \varrho(x_0, y) \quad \text{for } x, y \in A.$$

We then have

$$\|f(x) - f(y)\| = \varrho(x, y) \leqslant \varphi d(x, y) \quad \text{for all } x, y \in A.$$

Equipping X with the metric $\tilde{d}$ defined by $\tilde{d}(x, y) = \varphi\big(d(x, y)\big)$, we see that f: $(A, \tilde{d}) \to m(A)$ is a Lipschitz map. Thus, by 2.1, there is a map $\tilde{f}$: $(X, \tilde{d}) \to m(A)$ such that $\|\tilde{f}(x) - \tilde{f}(y)\| \leqslant \varphi\big(d(x, y)\big)$ for $x, y \in X$. Clearly, f is uniformly continuous with respect to d. The result follows from 2.2.

2.5. COROLLARY. *Let A be a subset of a metric space (X, d). Then any bounded metric ϱ on X uniformly equivalent to $d|A \times A$ can be extended to a metric $\bar{\varrho}$ on X which is uniformly equivalent to d on X.*

Proof. If ϱ is bounded, then

$$\lim_{t\to\infty} t^{-1}\omega_{\varrho/d}(t) = 0.$$

3. The unlimited uniform extension property for metrics.

3.1. Definition. Let d and ϱ be metrics on a set A uniformly equivalent to each other. We say that ϱ has the *unlimited uniform extension property* with respect to d if for every metric space (X, d) containing (A, d) isometrically there exists a metric $\bar{\varrho}$ on X which is uniformly equivalent to d on X and extends ϱ.

3.2. THEOREM. *Let d and ϱ be metrics uniformly equivalent to each other on a set A. Then ϱ has the unlimited uniform extension property with* *respect to d if and only if*

$$\limsup_{t\to\infty} t^{-1}\omega_{\varrho/d}(t) < \infty. \tag{6}$$

Proof. If (6) holds, then by 2.4 we infer that ϱ has the unlimited uniform extension property with respect to d.

Conversely, assume that (6) is not true and let $T\colon (A, d) \to m(A)$ be a map defined by

$$(Tx)y = \varrho(x, y) - \varrho(x_0, y) \quad \text{for } x, y \in A.$$

Then T is uniformly continuous and $\omega_{\varrho/d}(t)$ is a modulus of continuity of T. By a theorem of Aronszajn and Panitchpakdi [2] there exists a metric space $(X, \tilde{d})$ containing (A, d) isometrically and such that T cannot be extended to a uniformly continuous map $\tilde{T}\colon (X, \tilde{d}) \to m(A)$. It can easily be seen that ϱ cannot be extended to a metric $\bar{\varrho}$ on X which is uniformly equivalent to D and X.

3.3. Definition (Atsuji [3], [4], Ramer [17]). Let (X, d) be a metric space. A finite sequence $\{x_i\}_{i=0}^{n}$ of points of X is called an *ε-chain of length n* provided that $d(x_i, x_{i-1}) \leqslant \varepsilon$ for $i = 1, \ldots, n$. We say that (X, d) is *finitely chainable* if for every $\varepsilon > 0$ there exist finitely many points $a_1, \ldots, a_m$ of X and a positive integer n such that any point of X can be bounded by an ε-chain of length n with some of the points $a_1, \ldots, a_m$.

We say that (X, d) is *0^+-connected* if for any $\varepsilon > 0$ any two points of X can be bounded by an ε-chain in X. If (X, d) is 0^+-connected, we put for $\varepsilon > 0$

$$d_\varepsilon(x, y) = \inf\Big\{\sum_{i=1}^{n} d(x_i, x_{i-1}),\ x = x_0,\ x_n = y\colon \\ d(x_i, x_{i-1}) \leqslant \varepsilon \text{ for } i = 1, \ldots, n\Big\}.$$

Obviously, d_ε is a metric on X uniformly equivalent to d on X. We say that (X, d) is *uniformly 0^+-chainable* provided that, for any $\varepsilon > 0$, d_ε is Lipschitz equivalent to d.

3.4. Theorem. *Let (A, d) be a metric space. If (A, d) is either finitely chainable or uniformly 0^+-chainable, then every metric ϱ on A uniformly equivalent to d has the unlimited uniform extension property with respect to d.*

For the proof we apply 2.3 and the results of Atsuji [3], [4] and Ramer [17] concerning the extension of uniformly continuous functions.

3.5. Corollary. *Let (A, d) be a metric space. If (A, d) is either compact or convex in a sense of Menger* (1)*, then any metric ϱ on A which is uniformly equivalent to d has the unlimited uniform extension property with respect to d.*

(1) That is, every pair of points of A can be joined in A by an arc isometric to a segment of the real line (see [16]).

Let us note that 3.5 follows also from 2.5 and 2.4. (If (A, d) is convex, then the function is subadditive, and hence $\lim_{t\to\infty} t^{-1}\omega_{\varrho/d}(t) < \infty$.)

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