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ALGEBRAIC RESULTS CONCERNING GREEN'S $\mathscr{H}$-SLICES

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0. Introduction. To determine whether a subset A in a compact semigroup is an $\mathscr{H}(T)$-slice, for a closed set T, is important because, if so, then A is homeomorphic to a topological group according to the Schutzenberger-Wallace Theorem. One of the results of this paper reduces the investigation of $\mathscr{H}(T)$-slices in commutative semigroups to those subsets T which are subsemigroups and another result gives necessary and sufficient algebraic conditions for a set A to be an $\mathscr{H}(T)$-slice if T is a subsemigroup.

1. Preliminary material. Terminology and background results are presented in this section.

1.1. Definition. A (topological) *semigroup* S is a non-null Hausdorff space together with a continuous associative multiplication. Precisely, a semigroup is such a function m: $S\times S\to S$ that

(i) S is a non-null Hausdorff space,

(ii) m is continuous, and

(iii) m is associative; i.e., for each x, y, z in S,

$$m(x, m(y, z)) = m(m(x, y), z).$$

For brevity, the multiplication is ordinarily denoted by juxtaposition, that is, $m(x, y) = xy$. Moreover, it is common usage to say that a semigroup S is *compact* if S is a compact space and to say that a subset of S is *closed* if it is closed in a topological sense.

1.2. Definitions. For A, B subsets of a semigroup S, define $AB = \bigcup\{ab \mid a\in A \text{ and } b\in B\}$ and let $A^2 = AA$. If T is a subset of S, then T is a *subsemigroup* of S if and only if $T^2\subset T$.

1.3. Definitions. The empty set will be designated by $\square$. If X and Y are subsets of S, then $X^{(-1)}Y = \{w \text{ in } S;\ Xw\cap Y\neq\square\}$, $X^{[-1]}Y = \{w \text{ in } S;\ Xw\subset Y\}$, $YX^{-1)} = \{w \text{ in } S;\ wX\cap Y\neq\square\}$, and $YX^{[-1]} = \{w \text{ in } S;\ wX\subset Y\}$.

It is noted that if X is a singleton set, then $X^{(-1)}Y = X^{[-1]}Y$ and $YX^{(-1)} = YX^{[-1]}$. In addition, we will use the fact that if Y is closed, then $X^{[-1]}Y$ is closed.

1.4. Definition. Letting Y be a subset of S and Δ be the diagonal of $Y \times Y$, then an equivalence relation $R \subset Y \times Y$ is a *closed congruence* on Y if and only if $\Delta R \cup R\Delta \subset R$ and R is closed in $Y \times Y$ with respect to the relative topology.

It may be shown that if S is compact or discrete and if R is a closed congruence on S, then S/R is a semigroup and the canonical map from S to S/R is continuous [2].

1.5. Definition. If S is a semigroup and A and T are subsets of S, then one defines $L(A, T) = A \cup TA$, $R(A, T) = A \cup AT$ and $H(A, T) = R(A, T) \cap L(A, T)$. When the context clearly indicates which subset T is under consideration, then reference to T is usually omitted, that is, we write $L(A, T) = L(A)$, etc. Moreover, for $T \subset S$, one defines the Relative Green (equivalence) Relations, $\mathscr{L} = \{(x, y);\ x, y \in S$ and $L(x) = L(y)\}$, $\mathscr{R} = \{(x, y);\ R(x) = R(y)\}$ and $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$. For $x \in S$, we will let $H_x(T)$ denote the $\mathscr{H}(T)$-class (or slice) containing x; here again reference to T is omitted if the context is clear.

It is easy to verify that if $T^2 \subset T$, then $\mathscr{H} = \{(x, y);\ H(x) = H(y)\}$. Also, it is true that $H_w^{(-1)}H_w = H_w^{[-1]}H_w = w^{(-1)}H_w$ for any $w \in S$ (see [2]).

1.6. Definition. For any $A \subset S$ and $y \in S$, let us define $\mathscr{S}(A, y) = \{(u, v);\ u, v \in A^{[-1]}A$ and $yu = yv\}$ and $\mathscr{T}(A, y) = \{(u, v);\ u, v \in AA^{[-1]}$ and $uy = vy\}$.

It is well known that if a compact semigroup is algebraically a group, then it is a topological group, a result proved by M. Moriya [1]. Using this fact one may show that the Schutzenberger-Wallace Theorem follows (see [1]):

If S is compact or discrete, if T is a closed subset of S and if y is an element of S such that $\operatorname{card} H_y > 1$, then H_y is homeomorphic to the topological group, $y^{(-1)}H_y/\mathscr{S}(H_y, y)$, and the groups $y^{(-1)}H_y/\mathscr{S}(H_y, y)$ and $H_y y^{(-1)}/\mathscr{T}(H_y, y)$ are isomorphic.

The groups mentioned above, namely, $y^{(-1)}H_y/\mathscr{S}(H_y, y)$ and $H_y y^{(-1)}/\mathscr{T}(H_y, y)$, are commonly referred to as the Schutzenberger groups.

2. Results. In (2.3) necessary and sufficient algebraic conditions for a set A to be an $\mathscr{H}(T)$-slice, if T is a subsemigroup, are given. Consequently, in view of (2.5), a theorem which reduces the study of non-trivial $\mathscr{H}(T)$-slices in a commutative semigroup to those subsets T which are also subsemigroups, one may perceive (2.3) as an algebraic characterization of non-trivial $\mathscr{H}$-slices in commutative semigroups.

It is well known that a semigroup is a group if and only if it is an $\mathscr{H}$-slice so that, in particular, if a semigroup is not a group, then it is not an $\mathscr{H}$-slice (for any $T \subset S$). Semigroups which are not groups are not the only sets which fail to be $\mathscr{H}$-slices, as the following example indicates:

2.1. Example. Let S be a semigroup containing more than two elements with multiplication $xy = c$ for some fixed $c \in S$ and let A be any subset of S containing more than one element such that $c \notin A$. Clearly, A is not a semigroup and, since each element is its own $\mathscr{H}$-equivalence class (for any T) in such a semigroup, A is not an $\mathscr{H}$-slice.

The previous example also shows that the conditions $A^{[-1]}A = A^{(-1)}A$ and $AA^{[-1]} = AA^{(-1)}$ are not sufficient for a set A to be an $\mathscr{H}$-slice. It may be noted, as partially indicated in Section 1, that these are necessary conditions.

In general, what constitutes necessary and sufficient conditions for a subset of a semigroup to be an $\mathscr{H}$-slice for some T remains an open question; however, if $T^2 \subset T$ we can specify such algebraic conditions as indicated in the subsequent theorem:

2.2. Lemma. *Let $T \subset S$ and A be a non-empty subset of S and consider the following conditions*:

(1) *If a and b are distinct elements of A, then $b \in aT \cap Ta$.*

(2) *If $a \in A$ and $x \in S \setminus A$, then at least one of the following four sets is empty*:

$$T \cap a^{(-1)}x, \quad T \cap xa^{(-1)}, \quad T \cap x^{(-1)}a, \quad T \cap ax^{(-1)}.$$

Then:

(a) *if A is contained in an $\mathscr{H}(T)$-slice, then condition* (1) *holds.*

(b) *If condition* (1) *holds and $T^2 \subset T$, then $A \subset H_a(T)$ for $a \in A$.*

(c) *If condition* (2) *holds, then $H_a(T) \subset A$ for $a \in A$.*

(d) *If $H_a(T) \subset A$ for $a \in A$ and $T^2 \subset T$, then condition* (2) *is true.*

Proof. (a) Suppose $A \subset H_a(T)$ for $a \in A$. If $\operatorname{card} A = 1$, then condition (1) is satisfied vacuously; if $\operatorname{card} A > 1$, then condition (1) is immediate.

(b) If a and b are distinct elements of $A, T^2 \subset T$, and condition (1) holds, then $L(a) = a \cup Ta = tb \cup Ttb \subset Tb \subset L(b)$ for some $t \in T$ and, similarly, $L(b) \subset L(a)$ and $R(b) = R(a)$. Thus $H(a) = H(b)$ and, since $T^2 \subset T$, $(a, b) \in \mathscr{H}$.

(c) If $H_a(T) \not\subset A$, i.e., there exists an $x \in S \setminus A \cap H_a(T)$ so that $L(x) = L(a)$ and $R(x) = R(a)$, then the sets in condition (2) are all non-empty.

(d) If condition (2) is not true and $T^2 \subset T$, then for some $x \in S \setminus A$ and for some $a \in A$ all the sets in condition (2) are non-empty and $L(a)$

$= L(x)$ and $R(a) = R(x)$. Thus $H(a) = H(x)$ and, since $T^2 \subset T$, $x \in H_a(T)$ so that $H_a(T) \not\subset A$.

2.3. THEOREM. *Suppose $T^2 \subset T \subset S$. A non-empty subset A of S is an $\mathscr{H}(T)$-slice if and only if the following conditions hold:*

(1) *If a and b are distinct elements of A, then $b \in aT \cap Ta$.*

(2) *If $a \in A$ and $x \in S \setminus A$, then at least one of the following four sets is empty: $T \cap a^{(-1)}x$, $T \cap xa^{(-1)}$, $T \cap x^{(-1)}a$, $T \cap ax^{(-1)}$.*

Proof. In view of the lemma, this result is immediate.

If c is an element of a semigroup S such that $xy = c$ for all $x, y \in S$, if T is a subset of S and if $b \neq c$, then $\{b\} = H_b(T)$ and yet there exists no $t \in T$ such that $bt = b$. Hence, this example indicates that the word *distinct* may not be omitted from condition (1) in the 2.3 Theorem nor may it be removed from condition (1) as it applies in part (a) of (2.2).

If for $a \in S$ and $B \subset S$ we define ra: $B \to S$ by $ra(b) = ba$ and la: $B \to S$ by $la(b) = ab$, then it is possible to formulate (2.3) in functional notation:

2.3′. THEOREM. *Let $T^2 \subset T \subset S$. If the domain for the functions la and ra is T, then a non-empty subset A of S is an $\mathscr{H}(T)$-slice if and only if the following two conditions are satisfied:*

(1′) *$la[(la)^{-1}(A \setminus a)] = ra[(ra)^{-1}(A \setminus a)] = A \setminus a$ for each $a \in A$.*

(2′) *If $a \in A$ and $x \in S \setminus A$, then at least one of the following four sets is empty: $(la)^{-1}(x)$, $(ra)^{-1}(x)$, $(lx)^{-1}(a)$, $(rx)^{-1}(a)$.*

Proof. It suffices to show the equivalence of the conditions of the 2.3 and 2.3′ Theorems. Since $(la)^{-1}(x) = T \cap a^{(-1)}x$, it is evident that conditions (2) and (2′) are the same because in a similar manner equalities for the other three sets may be obtained; and so it remains to exhibit the equivalence of conditions (1) and (1′):

If $la[(la)^{-1}(A \setminus a)] = A \setminus a$ for all $a \in A$ and if b and c are distinct elements of A, then $b \in A \setminus c$ implies the existence of an element $t \in (lc)^{-1}(A \setminus c)$ such that $ct = b$. In a similar manner, $ra[(ra)^{-1}(A \setminus a)] = A \setminus a$ for all $a \in A$ implies that $b \in Tc$.

If a and b are distinct elements of A and if $b \in aT$, say $b = at$ for $t \in T$, then $t \in (la)^{-1}(b)$ and $la(t) = b$ so that $A \setminus a \subset la[(la)^{-1}(A \setminus a)]$. Since it is always the case that $la[(la)^{-1}(A \setminus a)] \subset A \setminus a$ and since it is easy to see that, in a similar fashion, $b \in Ta$ implies that $ra[(ra)^{-1}(A \setminus a)] = A \setminus a$ for all $a \in A$, we see that condition (1) implies condition (1′).

We conclude this section with a result which reduces the study of $\mathscr{H}$-slices in commutative semigroups to those sets T for which $T^2 \subset T$ and $T = \square$. Consequently, (2.3) takes on added significance since it deals with subsets T which are subsemigroups.

2.4. LEMMA. *If A is such a subset of a semigroup S that* $\operatorname{card} A > 1$, $A^{[-1]}A = A^{(-1)}A$, *condition* (1) *of* (2.2) *holds for some subset $T \subset S$ and A is normal in that T, that is, $xA = Ax$ for all $x \in T$, then A is an $\mathscr{H}(T')$-slice, where T' is the semigroup generated by $T \cap A^{[-1]}A$.*

Proof. For distinct elements $a, b \in A$ we have $[T \cap (ba^{(-1)} \cup a^{(-1)}b)] \subset T'$ so that condition (1) of (2.2) holds when we replace T by T'. Therefore, since T' is a semigroup, it follows from part (b) of (2.2) that $A \subset H_a(T')$, where $a \in A$. Now if $x \in H_a(T')$, then, because $T' \subset A^{[-1]}A$, we have that $x \cup xT' = a \cup aT' \subset A$ and so $H_a(T') \subset A$.

2.5. THEOREM. *If A is an $\mathscr{H}(T)$-slice which is normal in T and if* $\operatorname{card} A > 1$, *then A is an $\mathscr{H}(T')$-slice, where T' is the semigroup generated by $T \cap A^{[-1]}A$. As a result, in a commutative semigroup S to determine if a subset A of cardinality > 1 is an $\mathscr{H}$-slice for some T, it is sufficient to investigate the $\mathscr{H}$-slice decompositions yielded by the subsemigroups of S.*

Proof. This is an immediate corollary to (2.4) because the hypothesis that A is an $\mathscr{H}(T)$-slice implies that $A^{[-1]}A = A^{(-1)}A$ and that condition (1) of (2.2) holds.

Theorems (2.3) and (2.5) have several related topological questions. For example, what constitutes necessary and sufficient topological conditions for a set to be an $\mathscr{H}$-slice remains an open question (**P 669**). Also, in view of the Schutzenberger-Wallace Theorem, it would be of interest to see whether the semigroup T' in (2.5) is closed if T is closed (**P 670**).

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