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A UNIVERSAL NULL GRAPH WHOSE DOMAIN HAS POSITIVE MEASURE

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1. Introduction. An uncountable set $S \subseteq [0,1]$ such that every subset of S of Lebesgue measure 0 is necessarily countable was first demonstrated by Sierpiński with the aid of the continuum hypothesis (CH) (cf. [6], Proposition 20.1*). It is immediate that $\lambda^* S > 0$, where λ^* denotes outer Lebesgue measure. Such a set S is called a *Sierpiński set*.

A set $P \subseteq X$ (X separable metric) is said to be a *universal null set* if and only if every continuous measure (finite, non-negative, countably additive, defined on Borel sets) on X assigns P outer measure 0 if and only if every measure on P is totally atomic (cf. [9] for equivalence). By using CH, Lusin demonstrated an uncountable $P \subseteq [0,1]$ which, in addition to other properties, is universal null (cf. [6], Proposition 20.1). In fact, it is possible to construct an uncountable universal null set without axioms outside ZFC (see [9] and cf. [4], Theorem 1.2). The main purpose of this note is to describe, using CH, a Sierpiński set $S \subseteq [0,1]$ and a bijection $p: S \to S$ such that $p = p^{-1}$ and $\mathrm{graph}(p)$ is universal null in $[0,1]^2$. We then show that such a function p does not generally exist for a Sierpiński set $S \subseteq [0,1]$. The motivation for providing these examples was to answer the questions raised in [3]. The reader should notice that the existence of a universal null graph whose domain has positive measure follows (in ZFC) from Theorem A of [3].

2. The first example. We say that $H \subseteq [0,1]^2$ is *symmetric* provided that $(x,y) \in H$ implies $(y,x) \in H$. The line in $[0,1]^2$ containing the pairs $(0,1)$ and $(1,0)$ is denoted by L and the line containing $(0,0)$ and $(1,1)$ is denoted by M.

LEMMA. *If μ is a measure on $[0,1]^2$, then there exists a symmetric subset H of $[0,1]^2$ such that*

1. $\mu H = 0$;

2. *there exists a (relative) dense G_δ-subset h of L such that if $p \in h$, then the line in $[0,1]^2$ parallel to M and containing p lies in H.*

Proof. For μ on $[0,1]^2$, let μ' be the measure on L, $\mu' = \mu \circ \pi^{-1}$, where π is the projection of $[0,1]^2$ onto L parallel to M. There exists a (relative) dense G_δ-set $h \subseteq L$ such that $\mu' h = 0$. Furthermore, we can assume that h is symmetric. Let $H = \pi^{-1}(h)$.

THEOREM (CH). *There exist a Sierpiński set $S \subseteq [0,1]$ and a bijection $p\colon S \to S$ with $p = p^{-1}$ such that* graph(p) *is a universal null set in* $[0,1]^2$.

Proof. Index the continuous measures on $[0,1]^2$ by the countable ordinals $\langle \mu_a \rangle$. For each μ_a, let H_a be the set H from the Lemma for $\mu = \mu_a$. Index the G_δ-subsets of $[0,1]$ with λ-measure 0 by the countable ordinals $\langle G_a \rangle$.

At the ordinal 0, choose (x_0, y_0) in the set $H_0 \setminus \big(\pi_x^{-1}(G_0) \cup \pi_y^{-1}(G_0)\big)$, where π_x means the projection onto the x-axis and π_y onto the y-axis. It is possible to choose (x_0, y_0) since $\lambda G_0 = 0$ and H_0 contains a line parallel to M. Furthermore, (y_0, x_0) is also in the set since H_0 is symmetric, $x_0 \notin G_0$ and $y_0 \notin G_0$.

At the ordinal β, choose (x_β, y_β) in the set

$$\Big[\bigcap_{a\leqslant\beta} H_a\Big] \setminus \Big[\pi_x^{-1}\Big(\bigcup_{a\leqslant\beta} G_a \cup \bigcup_{a<\beta}\{x_a, y_a\}\Big) \cup \pi_y^{-1}\Big(\bigcup_{a\leqslant\beta} G_a \cup \bigcup_{a<\beta}\{x_a, y_a\}\Big)\Big].$$

This is possible since $\lambda\Big(\bigcup_{a\leqslant\beta} G_a \cup \bigcup_{a<\beta}\{x_a, y_a\}\Big) = 0$ and $\bigcap_{a\leqslant\beta} H_a$ contains a line parallel to M. Furthermore, (y_β, x_β) is also in the set.

Letting

$$S = \bigcup_{\beta<\omega_1}\{x_\beta, y_\beta\},$$

S is clearly an uncountable subset of $[0,1]$. However, if $B \subseteq S$ and $\lambda B = 0$, then $B \subseteq G_\beta$ for some β, so

$$B \subseteq \bigcup_{a<\beta}\{x_a, y_a\},$$

which makes B countable. Define $p\colon S \to S$ by $p(x_\beta) = y_\beta$ and $p(y_\beta) = x_\beta$ for each β. Then p is 1-1, onto, and $p = p^{-1}$. Furthermore, for each μ_β on $[0,1]^2$, only countably many points of graph(p) fail to lie in H_β, so $\mu_\beta^*(\text{graph}(p)) = 0$. Therefore, graph$(p)$ is universal null.

3. Another property of the first example. A set $P \subseteq X$ (X separable metric) has *strong measure* 0 means that if $\langle d_1, d_2, \ldots \rangle$ is any sequence of positive numbers, then there exists a sequence $\langle p_1, p_2, \ldots \rangle$ of points in P such that the collection of neighborhoods $\{N_{d_1}(p_1), N_{d_2}(p_2), \ldots\}$ covers P. In many parts of the literature, strong measure 0 is called *property C*, and universal null is called *property β* (see, e.g., [5]).

It is well known that C implies β (cf. [4], Theorem 1.3). It is also known that β does not imply C. This result is found in [5] by producing a β-set not having dimension 0, and showing that C-sets have dimension 0.

The result can also be deduced from [7] where a set is described having a property stronger than β whose square does not have property C. Since squares of β-sets are β, the conclusion follows. As mentioned in [4], p. 154-155, Sierpiński [8] constructed a β-set in $[0,1]$ and a continuous function $f\colon [0,1]\to[0,1]$ (also of bounded variation [1]) such that $f(\beta)$ is not β. As such a function f preserves property C, the conclusion follows. We obtain yet another example, since the graph of p is β, but is not C, else S would be C, and therefore $\lambda(S)=0$. Each of these examples assume CH.

Laver [4] showed that it is consistent with ZFC that property C is the same as countable. When coupled with the ZFC example of an uncountable β-set one obtains a β-set with which it is consistent that it is not C.

On the other hand, it follows from [2] that β does not imply C (in ZFC). That is, Grzegorek proved ([2], Corollary 2) that there exist subsets A and B of $[0,1]$ with $|A|=|B|$ and such that A is β and B is not β. Let f be a bijection from A onto B. The graph of f is β (since A is β) but not C (since B is not C). See also Corollary 3 of [3].

4. The second example.

THEOREM (CH). *There exists a Sierpiński set $S\subseteq[0,1]$ such that if $p\colon S\to S$ is a function, then* graph(p) *is not universal null.*

Proof. Let λ^2 denote Lebesgue measure on $[0,1]^2$. For $H\subseteq[0,1]^2$ and $M\subseteq[0,1]$, we have

$$H^M=\{y\colon \text{there exists } x\in M \text{ such that } (x,y)\in H\}.$$

Index the G_δ-subsets of $[0,1]$ with λ-measure 0, as before, by $\langle G_\alpha\rangle$. Index by $\langle H_\alpha\rangle$, $\alpha<\omega_1$, the Borel sets H in $[0,1]^2$ with $\lambda^2H=0$ and with the additional property that $\lambda(H^{\{x\}})=0$ for all $x\in[0,1]$.

Choose $x_0\in[0,1]$. At the level β, let

$$M=\bigcup_{\alpha<\beta}\{x_\alpha\},$$

and choose x_β from the set $[0,1]\setminus\big(\bigcup_{\alpha<\beta}G_\alpha\cup\bigcup_{\alpha<\beta}H^M\cup M\big)$. Letting

$$S=\bigcup_{\beta<\omega_1}\{x_\beta\},$$

S is clearly a Sierpiński set.

Now suppose that $p\colon S\to S$ is a function and let p^{-1} denote the inverse, although it need not be a function.

We claim that one of the following conditions is satisfied:

(1) p is essentially the identity function, hence graph(p) is essentially like S, and thus not universal null;

(2) an uncountable subset T of S is mapped to the same point, thus since $\lambda^* T > 0$, graph(p) is not universal null;

(3) $\lambda^{2*}(\text{graph}(p)) > 0$.

Supposing that none of the three conditions is met, there exists a Borel set $B \subseteq [0,1]^2$ such that B contains graph$(p) \cup$ graph(p^{-1}) and $\lambda^2 B = 0$ (since (3) fails). By Fubini's theorem (cf. [6], Theorem 14.2), there exists a Borel set $C \subseteq [0,1]$ such that $\lambda C = 0$ and if $x \in [0,1] \setminus C$, then $\lambda(B^{\{x\}}) = 0$. Now $B \setminus \pi_x^{-1}(C)$ is a Borel set in $[0,1]^2$, and since $C \cap S$ must be countable, this Borel set contains all but countably many points of graph(p), and since (2) fails, it must contain all but countably many points of graph(p^{-1}). Therefore, there exists a such that H_a contains graph$(p) \cup$ graph(p^{-1}). Since (1) and (2) fail, either graph(p) or graph(p^{-1}) contains a pair (x_β, x_γ) with $a < \beta < \gamma$. This violates the condition that $x_\gamma \notin H_a^M$, where M includes x_β. Thus the theorem is proved.

COROLLARY (CH). *There exists a countably generated, point separating σ-field $\boldsymbol{B}$ on a set S such that if p is a bijection from S onto S, then there exists a continuous probability measure on $\sigma(\boldsymbol{B} \cup p(\boldsymbol{B}))$.*

Proof. Combine the above example with the proof of Theorem A in [3].

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