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MEASURE PRESERVING ANALYTIC DIFFEOMORPHISMS OF COUNTABLE DENSE SETS IN C^n AND R^n

BY

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0. The classical Cantor theorem characterizing type η can be reformulated in the following way: for two arbitrary countable and dense subsets of R there exists a homeomorphism of R onto R establishing a 1-1 correspondence between them. L. E. J. Brouwer showed ([2], see also [3], p. 360, Problem 4.5.2) that, in the above theorem, R can be replaced by R^n. From the theorem of R. B. Bennett ([1], th. 3) it follows that R^n can be replaced by any manifold. In [4] P. Franklin proved that the homeomorphism in Cantor's theorem (on R) can be chosen to be an analytic function.

In this paper we show that for two arbitrary countable and dense subsets of C^n or of R^n for $n \geqslant 2$, where R^n is considered as a natural subspace of C^n, there exists an analytic diffeomorphism of C^n onto C^n establishing a 1-1 correspondence between these two sets. Additionally, the Jacobian of this diffeomorphism can be identically equal to one.

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1. By $B(z, r)$ we will denote the closed ball in C^n (with the Euclidean metric) of center $z \in C^n$ and radius $r > 0$. For $z = (z_1, z_2, \ldots, z_n) \in C^n$ let us define $S(z) = \sum z_i^2$. By id we will denote the identity function in C^n. Let us put $\theta = (0, 0, \ldots, 0) \in C^n$ (R^n). Let $K\colon C^n \to C^n$. By $DK(z)$ we will denote the Jacobi matrix of K at a point z. For the complex $n \times n$ matrices $[z_{ij}]$ we define the norm: $\|[z_{ij}]\| = \sum_{i,j} |z_{ij}|$.

THEOREM 1. *For two arbitrary sets A, B countable and dense in C^n, $n \geqslant 2$, there exists an analytic diffeomorphism F of C^n onto C^n such that $F(A) = B$ and* $\det DF(z) \equiv 1$.

Before we start the proof of Theorem 1 we will prove a few auxiliary lemmas.

Let P be a two-dimensional plane in $\mathbf{R}^n$. Let S_P denote the perpendicular projection (in $\mathbf{R}^n$) onto P. Let $o^P_{y,\varphi}$ denote the rotation on the plane P with center $y \in P$ and angle φ. We extend $o^P_{y,\varphi}$ to the rotation $O^P_{y,\varphi}$ in the whole space $\mathbf{R}^n$:

$$O^P_{y,\varphi}(x) = o^P_{y,\varphi}(S_P(x)) + (x - S_P(x)) \quad \text{for} \quad x \in \mathbf{R}^n.$$

LEMMA 1. *Let* $0 < \varphi < 2\pi$ *and* $\delta > 0$. *There exists* $\xi > 0$ *such that for* $s, t \in \mathbf{R}^n$, *where* $t \in B(s, \xi)$, *on any plane* P *containing the points* s, t *there lies a point* y *such that* $O^P_{y,\varphi}(s) = t$ *and* $y \in B(s, \delta)$.

Proof. It is enough to put $\xi < 2\delta \sin(\varphi/2)$.

Later on we will use the fact that each rotation in $\mathbf{R}^n$ with center θ can be expressed in the form $\exp(B)$, where B is a certain antisymmetric matrix, and conversely that each mapping of this form is a rotation in $\mathbf{R}^n$ with center θ.

Let $L = \{z^1, z^2, \ldots, z^m\} \subset C^n$, $\alpha \in C$, B be a real antisymmetric $n \times n$ matrix. Let for $z \in C^n$

$$W_L(z) = \prod_{i=1}^{m} (S(z^i) - S(z))$$

and

$$H(z) = H_{L,\alpha,B}(z) = \exp(\alpha W_L(z) B)(z). \tag{1}$$

It can be easily noticed that for $z^1, z^2, \ldots, z^m \in \mathbf{R}^n$ and $\alpha \in \mathbf{R}$ the function H acts as a rotation with center θ on each sphere $\{x \in \mathbf{R}^n: |x| = r\}$ and that $\det DH(x) = 1$ for each $x \in \mathbf{R}^n$. H is an analytic function of variable $z \in C^n$, and therefore we have $\det DH(z) = 1$ for each $z \in C^n$, where $z^1, z^2, \ldots, z^m \in \mathbf{R}^n$ and $\alpha \in \mathbf{R}$.

If z is fixed, then $\det DH(z) = f(z^1_1, \ldots, z^1_n, \ldots, z^m_1, \ldots, z^m_n, \alpha)$ is an analytic function of variables $z^1_1, \ldots, z^1_n, \ldots, z^m_1, \ldots, z^m_n, \alpha \in C$, and therefore the condition $\det DH(z) = 1$ holds for each $z \in C^n$ and for arbitrarily given $L = \{z^1, z^2, \ldots, z^m\} \subset C^n$ and $\alpha \in C$.

Let $z^1, z^2, \ldots, z^m \in \mathbf{R}^n$, $\alpha \in \mathbf{R}$. It can be easily noticed that

$$W_L(H_{L,\alpha,B}(x)) = W_L(x) \quad \text{for each} \quad x \in \mathbf{R}^n.$$

Hence

$$W_L(H_{L,\alpha,B}(z)) = W_L(z) \quad \text{for each } z \in C^n,$$

whence, using the same argumentation as above, we obtain

$$W_L(H_{L,\alpha,B}(z)) = W_L(z)$$

for each $z \in C^n$ and for arbitrarily given $L = \{z^1, z^2, \ldots, z^m\} \subset C^n$ and $\alpha \in C$. One can easily see now that $H_{L,\alpha,B}(H_{L,-\alpha,B}(z)) = z$ for each $z \in C^n$. Thus we obtain the following lemma.

LEMMA 2. *Each function $H_{L,\alpha,B}$ given by formula* (1) *is an analytic diffeomorphism of C^n onto C^n and* $\det H_{L,\alpha,B}(z) \equiv 1$.

LEMMA 3. *Let $f\colon C^n \to C^n$ be a continuous function. Let the inequality $|f(z)-z| < 1$ hold for each $z \in B(\theta, N+1)$. Then*

$$B(\theta, N) \subset f\big(B(\theta, N+1)\big).$$

Proof. Let $y \in B(\theta, N)$. Let us consider the function $g(z) = y-\big(f(z)-z\big)$. We have $g\big(B(\theta, N+1)\big) \subset B(\theta, N+1)$, and therefore from Brouwer's fixed point theorem it follows that there exists a point $z \in B(\theta, N+1)$ such that $z = g(z) = y-f(z)+z$. Thus $y = f(z)$.

LEMMA 4. *Let $J = \{c^1, c^2, \ldots, c^m\} \subset C^n$ and $s \in C^n \setminus J$, $\varepsilon > 0$, $N > 0$. Let D be a dense subset of C^n. Then there exists a diffeomorphism $G = (G_1, G_2, \ldots, G_n)$ of C^n onto C^n such that*: (i) $G(s) = d$ *for some* $d \in D$; (ii) $G(c^i) = c^i$, $i = 1, 2, \ldots, m$; (iii) $G_i\colon C^n \to C$ *is an analytic function*, $i = 1, 2, \ldots, n$; (iv) $\det DG(z) \equiv 1$; (v) $\|DG(z)-I\| < \varepsilon$, $|G(z)-z| < \varepsilon$, $\|DG^{-1}(z)-I\| < \varepsilon$ *and* $|G^{-1}(z)-z| < \varepsilon$ *for each* $z \in G(\theta, N)$, *where I denotes the $n \times n$ identity matrix.*

Proof. One can easily see that there exists a unitary transformation $V\colon C^n \to C^n$ such that $\min\{|S(V(c^i-s))|: i = 1, 2, \ldots, m\} = \eta > 0$. For a unitary transformation U by G_U we will denote the mapping $G_U(z) = U\big(V(z-s)\big)$.

Let $\zeta > 0$ be so small that if $\|B\| < \zeta$, where B is a real antisymmetric matrix, then for each mapping

$$K(z) = G_U^{-1}\big(H_{L,\alpha,B}(G_U(z)-y)+y\big), \tag{2}$$

where U is a unitary transformation, $y \in B(\theta, 1)$, $|\alpha| < (4/\eta)^m$, $L = \{G_U(c^i)-y\colon i = 1, 2, \ldots, m\}$, the inequalities

$$|K(z)-z| < \varepsilon, \qquad \|DK(z)-I\| < \varepsilon$$

hold for each $z \in B(\theta, N)$.

It follows from Lemma 2 that $\det K(z) \equiv 1$ for each K defined by (2).

Let $2\pi > \varphi > 0$ be so small that for each two-dimensional plane $P \subset \boldsymbol{R}^n$, $\theta \in P$, there exists an antisymmetric matrix B_φ^P such that $\exp(B_\varphi^P) = O_{\theta,\varphi}^P$ and $\|B\| < \zeta$.

Let $0 < \delta \leqslant 1$ be so small that the inequalities

$$\min\{|S\big(V(c^i-s)-z\big)|\colon i = 1, 2, \ldots, m\} > \eta/2, \qquad |S(z)| < \eta/4$$

hold for each $z \in B(\theta, \delta)$.

We choose ξ for δ and φ in accordance with Lemma 1. We can take a point $d \in B(s, \xi) \cap D$ and a unitary transformation U such that $G_U(d) \in \boldsymbol{R}^n$ and the inequality $\min\{|S(G_U(c^i)-z)|\colon i = 1, 2, \ldots, m\} > \eta/2$ holds for each $z \in B(\theta, \delta)$. By Lemma 1 there exists $y \in B(\theta, \delta) \cap \boldsymbol{R}^n$ such that $O_{\theta,\varphi}^P(-y) = G_U(d)-y$, where P is the two-dimensional plane containing the points θ, $G_U(d)-y$, $-y$. There exists a real antisymmetric matrix B_φ^P such that

$O^P_{\theta,\varphi} = \exp(B^P_\varphi)$ and $\|B^P_\varphi\| < \zeta$. We put $G(z) = K(z)$ for K given by (2), where y and U are as above and $L = \{G_U(c^i) - y\colon\ i = 1, 2, \ldots, m\}$, $\alpha = 1/W_L(-y)$, $B = B^P_\varphi$. Then $G(s) = d$. We have $-y \in B(\theta, \delta)$, whence $S(-y) < \eta/4$. Thus $|\alpha| < (4/\eta)^m$. One can easily see now that G has the desired properties.

Proof of Theorem 1.

Construction of F. Let $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$. We will construct a sequence of diffeomorphisms $G^i\colon C^n \to C^n$, $i = 0, 1, 2, \ldots$, such that the functions $F^i = G^i \circ G^{i-1} \circ \ldots \circ G^0$ converge almost uniformly to a function F satisfying the required conditions.

Let $G^0 = \mathrm{id}$. Let $r_0 = 1$ and $r_1 > 1$.

Let $\sigma > 0$. Suppose we have already constructed diffeomorphisms G^0, $G^1, \ldots, G^{2j}$ such that for $F^{2j} = G^{2j} \circ G^{2j-1} \circ \ldots \circ G^0$ we have $F^{2j}(a_i) = b_{v_i} \in B$, $F^{2j}(a_{u_i}) = b_i$, $a_{u_i} \in A$, $i = 1, 2, \ldots, j$, and $\det DF^{2j}(z) \equiv 1$. Let us assume that for $k = 0, 1, \ldots, 2j$ we have

$$|G^{2j} \circ G^{2j-1} \circ \ldots \circ G^k(z) - z| < \sigma \quad\text{and}\quad \|D(G^{2j} \circ G^{2j-1} \circ \ldots \circ G^k)(z) - I\| < \sigma$$

for each $z \in B(\theta, r_k)$, where

$$G^k \circ G^{k-1} \circ \ldots \circ G^0(B(\theta, k+1)) \subset B(\theta, r_{k+1})$$

and

$$G^{2j} \circ G^{2j-1} \circ \ldots \circ G^k(B(\theta, r_k)) \subset B(\theta, r_{2j+1}).$$

Let us also assume that $r_i > i$, $i = 0, 1, \ldots, 2j+1$.

Construction of G^{2j+1}. If $F^{2j}(a_{j+1}) \in B$, we put $G^{2j+1} = \mathrm{id}$ and we write $F^{2j}(a_{j+1}) = b_{v_{j+1}}$.

Let us assume that $F^{2j}(a_{j+1}) \notin B$. Let $\varepsilon > 0$ be so small that for each diffeomorphism G satisfying the inequalities (v) of Lemma 4 for $N = r_{2j+1}$, the inequalities

$$|G \circ G^{2j} \circ G^{2j-1} \circ \ldots \circ G^k(z) - z| < \sigma$$

and

$$\|D(G \circ G^{2j} \circ G^{2j-1} \circ \ldots \circ G^k(z) - I\| < \sigma$$

hold for each $z \in B(\theta, r_k)$, $k = 0, 1, \ldots, 2j$. We assume that $\varepsilon < \min(\sigma, 1/2^{2j+1})$. Let $J = \{b_1, b_2, \ldots, b_j, b_{v_1}, b_{v_2}, \ldots, b_{v_j}\}$, $s = F^{2j}(a_{j+1})$, $D = B$, $N = r_{2j+1}$. Let G be chosen for J, s, ε, N, D in accordance with Lemma 4. Let us put $G(s) = b_{v_{j+1}}$. We put $G^{2j+1} = G$ and $F^{2j+1} = G^{2j+1} \circ G^{2j} \circ \ldots \circ G^0$.

From the properties of G^{2j+1} and of $G^0, G^1, \ldots, G^{2j}$ it follows that

$$|G^{2j+1} \circ G^{2j} \circ \ldots \circ G^k(z) - z| < \sigma$$

and

$$\|D(G^{2j+1} \circ G^{2j} \circ \ldots \circ G^k)(z) - I\| < \sigma$$

for each $z \in B(\theta, r_k)$, $k = 0, 1, \ldots, 2j+1$. Also $F^{2j+1}(a_i) = b_{v_i}$, $i = 1, 2, \ldots, j+1$, $F^{2j+1}(a_{u_i}) = b_i$, $i = 1, 2, \ldots, j$, and $\det DF^{2j+1} \equiv 1$.

Construction of G^{2j+2}. If $b_{j+1} \in F^{2j+1}(A)$, we put $G^{2j+2} = \mathrm{id}$ and we write $a_{u_{j+1}} = (F^{2j+1})^{-1}(b_{j+1})$.

Let us assume that $b_{j+1} \notin F^{2j+1}(A)$. Let $r_{2j+2} > 2j+2$ be so large that

$$G^{2j+1} \circ G^{2j} \circ \ldots \circ G^k(B(\theta, r_k)) \subset B(\theta, r_{2j+2}), \quad k = 0, 1, \ldots, 2j+1,$$

and

$$G^{2j+1} \circ G^{2j} \circ \ldots \circ G^0(B(\theta, 2j+2)) \subset B(\theta, r_{2j+2}).$$

Let $\varepsilon < \min(\sigma, 1/2^{2j+2})$ be so small that for each diffeomorphism G satisfying the inequalities (v) of Lemma 4 for $N = r_{2j+2}$ the inequalities

$$|G \circ G^{2j+1} \circ G^{2j} \circ \ldots \circ G^k(z) - z| < \sigma$$

and

$$\|D(G \circ G^{2j+1} \circ G^{2j} \circ \ldots \circ G^k)(z) - I\| < \sigma$$

hold for each $z \in B(\theta, r_k)$, $k = 0, 1, \ldots, 2j+1$. Let

$J = \{b_1, b_2, \ldots, b_j, b_{v_1}, b_{v_2}, \ldots, b_{v_{j+1}}\}$, $s = b_{j+1}$, $D = F^{(2j+1)}(A)$, $N = r_{2j+2}$.

Let G be chosen for J, s, ε, N, D in accordance with Lemma 4. Let $d = F^{2j+1}(a_{u_{j+1}})$. We put

$$G^{2j+2} = G^{-1}, \quad F^{2j+2} = G^{2j+2} \circ G^{2j+1} \circ \ldots \circ G^0.$$

Let us also take $r_{2j+3} > 2j+3$ so large that

$$G^{2j+2} \circ G^{2j+1} \circ \ldots \circ G^k(B(\theta, r_k)) \subset B(\theta, r_{2j+3}), \quad k = 0, 1, \ldots, 2j+2,$$

and

$$G^{2j+2} \circ G^{2j+1} \circ \ldots \circ G^0(B(\theta, 2j+3)) \subset B(\theta, r_{2j+3}).$$

It follows from the construction that the sequence $\{F^k\}_{k=0}^{\infty}$ is almost uniformly convergent and for $F = \lim_{k \to \infty} F^k$ we have $F(A) = B$. By the Weierstrass theorem $F\colon C^n \to C^n$ is an analytic function and $\det DF(z) \equiv 1$. It remains to show that F is a diffeomorphism of C^n onto C^n. Let us take $\sigma < 1/4^n$. Let us put

$$\Phi^k = \lim_{j \to \infty} G^j \circ G^{j-1} \circ \ldots \circ G^k, \quad k = 0, 1, 2, \ldots$$

By the Weierstrass theorem we obtain from the construction of F the inequality

$$\|D\Phi^k(z) - I\| < 1/4^n \quad \text{for each } z \in B(\theta, r_k),\ k = 0, 1, 2, \ldots$$

Let us put $C^n = R^{2n}$ by taking $(z_1, \ldots, z_n) = (\mathrm{Re}\, z_1, \mathrm{Im}\, z_1, \ldots, \mathrm{Re}\, z_n, \mathrm{Im}\, z_n)$.

Using the Lagrange mean value theorem for $\mathbf{R}^{2n}$ one can easily check that

$$|\Phi^k(z)-\Phi^k(y)| > \frac{1}{4^n}|z-y| \quad \text{for} \quad z, y \in B(\theta, r_k),\ k = 0, 1, 2, \ldots,$$

whence Φ^k is injective on $B(\theta, r_k)$. Hence F is injective on $B(\theta, k)$ because $G^{k-1} \circ G^{k-2} \circ \ldots \circ G^0(B(\theta, k)) \subset B(\theta, r_k)$ and because the functions G^i are diffeomorphisms of C^n onto C^n. Thus F is injective on C^n.

We have $F(C^n) = \Phi^k(C^n) \supset \Phi^k(B(\theta, r_k))$, $k = 0, 1, 2, \ldots$ From the construction of F it follows that $|\Phi^k(z)-z| \leqslant \sigma$ for each $z \in B(\theta, r_k)$, $k = 0, 1, 2, \ldots$, whence by Lemma 3

$$B(\theta, r_k-1) \subset \Phi^k(B(\theta, r_k)).$$

Hence $B(\theta, k-1) \subset F(C^n)$. Thus $F(C^n) = C^n$. This completes the proof of Theorem 1.

If we restrict our considerations to $\mathbf{R}^n$ regarded as a natural subspace of C^n, by the same methods as above we are able to prove the following theorem.

THEOREM 2. *For two arbitrary sets A, B countable and dense in $\mathbf{R}^n$, $n \geqslant 2$, there exists an analytic diffeomorphism F of C^n onto C^n such that $F(A) = B$ and* $\det DF(z) \equiv 1$.

Remark 1. If we put as above $C^n = \mathbf{R}^{2n}$, then the condition $\det DF(z) \equiv 1$ of Theorems 1 and 2 implies that F preserves the Lebesgue measure in $\mathbf{R}^{2n}$. Additionally, in Theorem 2 F preserves the Lebesgue measure in $\mathbf{R}^n \subset C^n$.

Remark 2. Theorems 1 and 2 do not hold in the one-dimensional case because only the non-constant linear functions are analytic diffeomorphisms of C onto C.

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