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A CHARACTERIZATION OF FINITELY IRREDUCIBLE CONTINUA

BY

T. MAĆKOWIAK (WROCŁAW)

It is proved that a metric continuum X is irreducible about a finite set if and only if X is not the union of a countable monotonic collection of proper subcontinua. This equivalence is a solution of Fugate's question posed in the University of Houston Mathematics Problem Book (Problem 113).

All considered spaces are metric continua. Denote the collection of all nonempty subcontinua of X by $C(X)$. Put

$$C_0(X) = \{K \in C(X)\colon\ K \neq X\},$$

$$S(X) = \{A \subset X\colon \text{ if } A \subset K \in C(X), \text{ then } K = X\},$$

$$I(X) = \{A \subset X\colon \text{ there exists } a \in X \text{ such that } A \cup \{a\} \in S(X)\}.$$

By a slight modification of the proof of Théorème XIX in [3], p. 270 (cf. [4], Theorem 4, p. 192), it has been proved in [5], Theorem 3, p. 336, that

(1) *$A \notin I(X)$ if and only if there are $P, R \in C_0(X)$ such that*

$$X = P \cup R \quad \text{and} \quad A \subset P \cap R.$$

If $\{a, b\} \in S(X)$, then the continuum X is called *irreducible* (*between* a *and* b). If there is a finite set in X belonging to $S(X)$, then X is called *finitely irreducible* (or *irreducible about a finite set*). If K is a minimal subcontinuum of X intersecting both A and B, where A and B are closed, nonempty and disjoint subsets of X, then we say that K is *irreducible between A and B*. The following is known (see [4], Theorems 2 and 5, p. 220):

(2) *If K is a subcontinuum of X which is irreducible between A and B, then the set $K \setminus (A \cup B)$ is connected and dense in K and $\{a, b\} \in S(K)$ for $a \in A \cap K$ and $b \in B \cap K$.*

We have

(3) *If $\{K, L, M\} \subset C_0(X)$, $K \cup L \in S(X)$, $M \cap K \neq \emptyset \neq M \cap L$, then the set $X \setminus (K \cup L)$ is connected and contained in M.*

We can assume that $K \cap L = \emptyset$. Let N be an irreducible continuum between K and L in M. Then $X = K \cup N \cup L$ because $K \cup L \in S(X)$. Moreover, the set

$$X \setminus (K \cup L) = N \setminus (K \cup L)$$

is connected by (2), i.e., (3) holds.

(4) *If B is a subset of X, $A_n \in C_0(X)$, $A_n \cup B \in S(X)$, $A_{n+1} \subset A_n$ for $n = 1, 2, \ldots$, and*

$$A = \bigcap_{n=1}^{\infty} A_n,$$

then $A \cup B \in S(X)$.

In fact, if K is a subcontinuum of X containing $A \cup B$, then $X = K \cup A_n$ for $n = 1, 2, \ldots$ because $K \cup A_n \in C(X)$ and $A_n \cup B \in S(X)$ for $n = 1, 2, \ldots$ Hence $K = K \cup A = X$.

For $A \subset X$ let

$$I(A, X) = \{B \in C_0(X) \colon A \cup B \in S(X)\}$$

and

$$I_0(A, X) = \{B \in I(A, X) \colon \text{if } C \in C_0(B), \text{ then } C \notin I(A, X)\}.$$

By Brouwer's reduction theorem and (4) we obtain

(5) *If $I(A, X) \neq \emptyset$, then $I_0(A, X) \neq \emptyset$.*

A continuum X is called a *Θ-continuum* provided for each $K \in C_0(X)$ the set $X \setminus K$ has finitely many components. By an easy induction we obtain (see [6], (2.5), cf. [1], Theorem 3.4)

(6) *If X is a Θ-continuum and $\{K_1, K_2, \ldots, K_n\} \subset C_0(X)$, then the set $X \setminus (K_1 \cup K_2 \cup \ldots \cup K_n)$ has a finite number of components.*

We have

(7) *If X is a Θ-continuum, $N, M, R \in C_0(X)$, $a \in M \subset X \setminus N$, $X = N \cup R \cup M$ and R is irreducible between N and $\{a\}$, then $X \setminus (M \cup N)$ contains exactly one component C such that*

$$\mathrm{cl}(C) \cap N \neq \emptyset \neq \mathrm{cl}(C) \cap M.$$

Moreover, the set $\mathrm{cl}\, C$ *is an irreducible continuum between M and N.*

According to (6) the set $X \setminus (M \cup N)$ has a finite number of components. Therefore, the component C of $X \setminus (M \cup N)$ such that

$$\mathrm{cl}(C) \cap N \neq \emptyset \neq \mathrm{cl}(C) \cap M$$

is open. The set $\mathrm{cl}\, C$ is a subcontinuum of R containing a point from $N \cap R$. Therefore, the set $R \setminus \mathrm{cl}\, C$ is connected by (2) and Theorem 3 in [4], p. 193.

Since

$$a \in \mathrm{cl}(R \setminus \mathrm{cl}\, C) \subset R \setminus C$$

(C is open in R), we conclude that $\mathrm{cl}(R \setminus \mathrm{cl}\, C) \cap N = \emptyset$. But

$$X \setminus (M \cup N \cup C) \subset R \setminus \mathrm{cl}\, C \subset X \setminus N.$$

Therefore, no component of $X \setminus (M \cup N)$ different from C intersects N. The set C is open and connected and $C \subset R$, so

$$\mathrm{cl}\, C = \mathrm{cl}(\mathrm{int}(\mathrm{cl}\, C)),$$

i.e., $\mathrm{cl}\, C$ is a closed connected domain in R. Hence $\mathrm{cl}\, C$ is irreducible between every point of $N \cap \mathrm{cl}\, C$ and every point of the boundary of $\mathrm{cl}\, C$ in R by Theorem 1 in [4], p. 195, so (7) holds.

We write $X \in \mathscr{F}$ provided that X is a continuum which is not the union of a countable monotonic collection of proper subcontinua. Observe first that

(8) *If a continuum X is finitely irreducible, then $X \in \mathscr{F}$.*

Suppose X is irreducible about the set $\{a_1, a_2, \ldots, a_n\}$ and

$$X = \bigcup_{i=1}^{\infty} K_i,$$

where $K_i \subset K_{i+1}$ and $K_i \in C_0(X)$ for $i = 1, 2, \ldots$ Then $\{a_1, a_2, \ldots, a_n\} \subset K_j$ for some j. Thus $K_j = X$, because $\{a_1, a_2, \ldots, a_n\} \in S(X)$; a contradiction.

(9) *If $X \in \mathscr{F}$, then X is a Θ-continuum.*

Let $K \in C_0(X)$ and suppose that the set $X \setminus K$ has infinitely many components. The induction gives us the sets A_j^i for $i = 0, 1$ and $j = 1, 2, \ldots$ such that

$$X \setminus K = A_1^0 \cup A_1^1, \quad A_j^0 = A_{j+1}^0 \cup A_{j+1}^1,$$

the sets A_j^0 and A_j^1 are nonempty and separated and A_j^0 contains infinitely many components of $X \setminus K$ for $j = 1, 2, \ldots$ The sets $K \cup A_j^i$ are continua. Since

$$K \cup A_{j+1}^0 \subset K \cup A_j^0,$$

the set

$$K' = K \cup \bigcap_{j=1}^{\infty} A_j^0$$

is a continuum. Therefore, the sets

$$R_k = K' \cup \bigcup_{j=1}^{k} A_j^1$$

are continua. Since

$$R_k \subset R_{k+1}, \quad X = \bigcup_{k=1}^{\infty} R_k \quad \text{and} \quad R_k \in C_0(X),$$

we infer that $X \notin \mathscr{F}$, a contradiction.

(10) *If U is an open and connected subset of $X \in \mathscr{F}$, then* $\operatorname{cl} U \in \mathscr{F}$.

The set $X \setminus \operatorname{cl} U$ has a finite number of components $C_1, \ldots, C_n$ by (9). Suppose

$$\operatorname{cl} U = \bigcup_{i=1}^{\infty} K_i,$$

where $K_i \subset K_{i+1}$ and $K_i \in C_0(\operatorname{cl} U)$ for $i = 1, 2, \ldots$ Since

$$\operatorname{cl} C_i \cap \operatorname{cl} U \neq \emptyset \quad \text{for } i = 1, \ldots, n,$$

we can assume that $K_m \cap \operatorname{cl} C_i \neq \emptyset$ for $i = 1, \ldots, n$. Put

$$R_j = K_{m+j} \cup \operatorname{cl} C_1 \cup \ldots \cup \operatorname{cl} C_n \quad \text{for } j = 1, 2, \ldots$$

The sets R_j are continua, $R_{j+1} \supset R_j$ and

$$X = \bigcup_{j=1}^{\infty} R_j.$$

Since $X \in \mathscr{F}$, we infer that $R_j = X$ for some j. Since

$$U \cap (\operatorname{cl} C_1 \cup \ldots \cup \operatorname{cl} C_n) = \emptyset,$$

we obtain $U \subset K_{m+j}$; thus $\operatorname{cl} U = K_{m+j}$; a contradiction, because $K_{m+j} \in C_0(\operatorname{cl} U)$.

(11) *If $X \in \mathscr{F}$, $A \in I_0(B, X)$, then $A \in \mathscr{F}$.*

Suppose

$$A = \bigcup_{i=1}^{\infty} K_i,$$

where $K_i \subset K_{i+1}$ and $K_i \in C_0(A)$ for $i = 1, 2, \ldots$ Since the set $X \setminus A$ has a finite number of components (cf. (9)), we can assume that K_1 intersects the closure of every component of $X \setminus A$. Therefore, the sets $\operatorname{cl}(X \setminus A) \cup K_i$ are continua,

$$\operatorname{cl}(X \setminus A) \cup K_i \subset \operatorname{cl}(X \setminus A) \cup K_{i+1}$$

and

$$X = \bigcup_{j=1}^{\infty} (\operatorname{cl}(X \setminus A) \cup K_j).$$

Since $X \in \mathscr{F}$, we infer that $\operatorname{cl}(X \setminus A) \cup K_j = X$ for some j. If L is a continuum containing the set $K_j \cup B$, then $L \cup A = X$, because $A \cup B \in S(X)$. Therefore,

$\mathrm{cl}(X \setminus A) \subset L$. Thus $L = L \cup K_j = X$, i.e., $K_j \in I(B, X)$. But $K_j \subset A \in I_0(B, X)$ implies $A = K_j$, a contradiction because $K_j \in C_0(A)$.

(12) *If* $X \in \mathscr{F}$, $K \in C_0(X)$ *and for each* $P \in C_0(X)$ *with* $K \subset P$ *we have* $K \cap \mathrm{cl}(X \setminus P) \neq \emptyset$, *then there is* L *such that*

$$K \subset L \in I(X) \cap C_0(X).$$

By induction and (1) we construct P_i, $R_i \in C_0(X)$ such that

$$X = P_i \cup R_i, \qquad K \subset P_1 \cap R_1, \qquad R_i \subset P_{i+1} \cap R_{i+1}$$

and the set $P_i \setminus R_i$ is connected for $i = 1, 2, \dots$ In fact, if $K \notin I(X)$, by (1), we find P_1, $R_1 \in C_0(X)$ with $X = P_1 \cup R_1$ and $K \subset P_1 \cap R_1$. According to (9) we may assume that $P_1 \setminus R_1$ is connected (we add all components of $P_1 \setminus R_1$ except one to R_1). In the same manner we find P_{i+1} and R_{i+1} if P_i and R_i are defined and $R_i \notin I(X)$. Since

$$P_{i+1} \setminus R_{i+1} \subset P_i \setminus R_i$$

and the sets $P_i \setminus R_i$ are connected, we find that the set P_0 defined by

$$P_0 = \bigcap_{i=1}^{\infty} \mathrm{cl}(P_i \setminus R_i)$$

is connected. Moreover, $P_0 \cap K \neq \emptyset$ (otherwise, $\mathrm{cl}(P_i \setminus R_i) \cap K = \mathrm{cl}(X \setminus R_i) \cap K = \emptyset$ for some i, a contradiction). Therefore, the sets $P_0 \cup R_i$ are continua. Since

$$X = P_0 \cup \bigcup_{i=1}^{\infty} R_i \quad \text{and} \quad P_0 \cup R_i \subset P_0 \cup R_{i+1}$$

for $i = 1, 2, \dots$, we infer that $P_0 \cup R_i = X$ for some i by assumption. Hence

$$\mathrm{cl}(P_{i+1} \setminus R_{i+1}) \cup R_i = P_i \cup R_i;$$

thus $P_i \setminus R_i \subset \mathrm{cl}(P_{i+1} \setminus R_{i+1}) \subset P_{i+1}$. But $R_i \subset P_{i+1}$, thereby

$$X = P_i \cup R_i = P_{i+1},$$

a contradiction.

(13) *If* $A \in I(X) \cap C_0(X)$ *and for each* $R \in C_0(X)$ *with* $A \subset R$ *we have* $A \cap \mathrm{cl}(X \setminus R) \neq \emptyset$, *then* $\mathrm{cl}(X \setminus A)$ *is an indecomposable continuum.*

In fact, suppose that

$$\mathrm{cl}(X \setminus A) = P \cup R, \quad \text{where } P, R \subset C_0(\mathrm{cl}(X \setminus A))$$

(the set $\mathrm{cl}(X \setminus A)$ is a continuum by (3)). According to (3) we can assume that $P \cap A = \emptyset$ and $R \cap A \neq \emptyset$. Then $A \cup R \in C_0(X)$ and

$$A \cap \mathrm{cl}(X \setminus (A \cup R)) \subset A \cap P = \emptyset,$$

a contradiction.

A subcontinuum K of X is called *terminal* if every subcontinuum of X which intersects K and $X \setminus K$ contains K. A continuum X is *colocally connected* at K if for every open neighbourhood U of K in X there exists an open set V such that $K \subset V \subset U$ and the set $X \setminus V$ is connected. The following is known (see [2] for the proof; cf. [6], (5.1)):

(14) *If $K \in C_0(X)$ and for each $L \in C_0(X)$ with $K \subset L$ there is $M \in C_0(X)$ such that $L \subset \operatorname{int} M$, then there is a terminal subcontinuum N of X such that $K \cap N = \emptyset$ and X is colocally connected at N.*

We have

(15) *If $X \in \mathcal{F}$ and X is colocally connected at $N \in C_0(X)$, then there is $M \in C_0(X \setminus N)$ such that $N \cup M \in S(X)$.*

Fix $a \in X \setminus N$ and let R be an irreducible continuum between N and $\{a\}$. Let V_i be open sets such that $a \in X \setminus V_i$, $X \setminus V_i$ is connected, $V_{i+1} \subset V_i$ for $i = 1, 2, \ldots$ and

$$N = \bigcap_{j=1}^{\infty} V_j.$$

The sets $N \cup R \cup (X \setminus V_i)$ are continua,

$$N \cup R \cup (X \setminus V_i) \subset N \cup R \cup (X \setminus V_{i+1}) \quad \text{for } i = 1, 2, \ldots$$

and

$$X = N \cup R \cup \bigcup_{j=1}^{\infty} (X \setminus V_j).$$

Since $X \in \mathcal{F}$, we infer that $X = N \cup R \cup (X \setminus V_j)$ for some j. Let C_j be a component of $V_j \setminus N$ such that

$$N \cap \operatorname{cl} C_j \neq \emptyset \neq (X \setminus V_j) \cap \operatorname{cl} C_j.$$

By (7) and (9), the set

$$M = (X \setminus V_j) \cup \big(V_j \setminus (N \cup C_j)\big)$$

is a continuum, $M \subset X \setminus N$ and $\operatorname{cl} C_j$ is irreducible between N and M; hence $N \cup M \in S(X)$.

(16) *If $X \in \mathcal{F}$, $K \in C_0(X)$, then there is N such that*

$$K \subset N \in I(X) \cap C_0(X).$$

In fact, according to (12) we can assume that for each $L \in C_0(X)$ with $K \subset L$ there is $M \in C_0(X)$ such that $L \subset \operatorname{int} M$. By (14) there is a terminal continuum P in X such that $K \cap P = \emptyset$ and X is colocally connected at P. It follows from (15) that there is $N \in C_0(X \setminus P)$ such that $P \cup N \in S(X)$ and $K \subset N$. Since P is terminal, we infer that $\{a\} \cup N \in S(X)$ for each $a \in P$, i.e., $N \in I(X)$.

(17) *If* $X \in \mathscr{F}$, *A is an indecomposable subcontinuum of X with the nonempty interior*, $L \in C_0(X)$ *and L intersects every composant of A, then* $A \subset L$.

Let $C_1, C_2, \ldots, C_n$ be components of $X \setminus A$ (cf. (9)). Let K_i be a proper subcontinuum of A intersecting simultaneously $\operatorname{cl} C_i$ and L. The set $(\operatorname{int} A) \setminus (K_1 \cup \ldots \cup K_n)$ is open and dense in A. We can assume that $a \in (\operatorname{int} A) \setminus N$, where

$$N = \operatorname{cl} C_1 \cup \ldots \cup \operatorname{cl} C_n \cup K_1 \cup \ldots \cup K_n \cup L.$$

Let $M_1, M_2, \ldots$ be a sequence of subcontinua of A such that $a \in M_1$, $M_1 \cap L \neq \emptyset$, $M_i \subset M_{i+1}$ and $\bigcup_{i=1}^{\infty} M_i$ is a composant of A. Let $K(a, 1/m)$ denote a ball around a with radius $1/m$. We can assume that

$$K(a, 1/m) \cap N = \emptyset \quad \text{for } m = 1, 2, \ldots$$

Denote the component of $X \setminus K(a, 1/m)$ containing N by N_m. The sets $N_m \cup M_m$ are continua,

$$N_m \cup M_m \subset N_{m+1} \cup M_{m+1} \quad \text{for } m = 1, 2, \ldots$$

and

$$X = \bigcup_{m=1}^{\infty} (N_m \cup M_m).$$

Since $X \in \mathscr{F}$, we infer that $X = N_k \cup M_k$ for some positive integer k. Then $K(a, 1/k) \subset M_k$, but M_k is a proper subcontinuum of A which is indecomposable, a contradiction.

(18) *If* $X \in \mathscr{F}$, *A is an indecomposable subcontinuum of X with the nonempty interior, then A contains a composant C such that* $C \subset \operatorname{int} A$.

It suffices to show, by (9), that if K is a component of $X \setminus A$, then

$$N = \bigcup \{C\colon C \text{ is a composant of } A \text{ and } C \cap \operatorname{cl} K \neq \emptyset\}$$

is of the first category. According to (17) there is a composant C_0 of A such that $C_0 \cap \operatorname{cl} K = \emptyset$. Fix $a \in C_0$ and let $K(a, 1/m)$ be a ball around a with radius $1/m$. Take

$$B_m = \bigcup \{L\colon L \text{ is a component of } A \setminus K(a, 1/m) \text{ and } L \cap \operatorname{cl} K \neq \emptyset\}.$$

The set B_m is closed in A and $B_m \cap C_0 = \emptyset$. Therefore, B_m is nowhere dense in A. But

$$N = \bigcup_{m=1}^{\infty} B_m,$$

so N is of the first category in A.

(19) *If* $X \in \mathscr{F}$ *and* A *is an indecomposable subcontinuum of* X *with the nonempty interior, then there is a point* $a \in A$ *such that* $A \subset L$ *provided*

$$a \in L \in C(X) \quad \text{and} \quad L \cap (X \setminus \operatorname{int} A) \neq \emptyset .$$

In fact, let C be a composant of A such that $C \subset \operatorname{int} A$ and $a \in C$ (cf. (18)). If K is a subcontinuum of L irreducible between a and $X \setminus \operatorname{int} A$, then the set $K \cap \operatorname{int} A$ is connected and dense in K by (2). Thereby $K \subset A$. If $A \neq K$, then $K \subset C$, and therefore $K \cap (X \setminus \operatorname{int} A) = \emptyset$, a contradiction. This implies $A = K \subset L$.

(20) *If* K *is an irreducible continuum in* X *and* U *is an open connected subset of* X *such that* $U \subset K$, *then* $\operatorname{cl} U$ *is an irreducible continuum.*

Indeed, let $\{a, b\} \in S(K)$. The set $\operatorname{cl} U$ is a closed connected domain in K. If $\{a, b\} \cap \operatorname{cl} U \neq \emptyset$, then the continuum $\operatorname{cl} U$ is irreducible by Theorem 1 in [4], p. 195. Assume $\{a, b\} \cap \operatorname{cl} U = \emptyset$. According to Theorem 3 in [4], p. 193, the set $K \setminus \operatorname{cl} U$ is the union of two open connected sets V and W in K, one of which contains a and the other contains b. Let $a \in V$; then $V \cup \operatorname{cl} U$ is a closed connected domain in K containing a. By Theorem 1 in [4], p. 195, the continuum $V \cup \operatorname{cl} U$ is irreducible between a and some point $c \in (\operatorname{cl} U) \setminus V$. The set $\operatorname{cl} U$ is a closed connected domain in $V \cup \operatorname{cl} U$ containing c. Therefore, once again by Theorem 1 in [4], p. 195, the continuum $\operatorname{cl} U$ is irreducible.

(21) *If* U *is an open connected subset of* $X \in \mathscr{F}$ *and* $\operatorname{cl} U$ *is an irreducible continuum, then there is a finite set* $A \subset X$ *such that* $U \subset M$ *provided* $A \subset M \in C(X)$.

Let $C_1, \ldots, C_m$ be the closures of components of $X \setminus \operatorname{cl} U$ (cf. (9)) and let $K = \operatorname{cl} U$, $\{a, b\} \in S(K)$. It follows from Theorems 2 and 4 in [4], pp. 195–196, that if $\mathscr{D}$ is the family of all closed connected domains containing the point a (in K) and besides $\emptyset \in \mathscr{D}$, then $\mathscr{D}$ can be indexed by the elements of a closed set $J \subset [0, 1]$ in such a manner that

$$(s < t) \Leftrightarrow (D_s \subset \operatorname{int}_K D_t) \text{ and } \emptyset \neq D_t \neq K \text{ for } t \in J \setminus \{0, 1\}.$$

Put $E(D) = \operatorname{cl}(K \setminus D)$ and $\mathscr{E} = \{E(D): D \in \mathscr{D}\}$. By Theorem 5 in [4], p. 196, the collection $\mathscr{E}$ is the family of all closed connected domains in K containing the point b augmented by the empty set and $E(E(D)) = D$ for $D \in \mathscr{D} \cup \mathscr{E}$.

It follows from Theorem 6 in [4], p. 197, that

$$\operatorname{bd}_K(D_t) = \operatorname{bd}_K(E_t) = D_t \cap E_t .$$

Therefore, since U is open, dense and connected in K, we infer that

(21.1) $D_t \cap E_t \cap U \neq \emptyset$ *for* $t \in J \setminus \{0, 1\}$.

Fix $i = 1, 2, \ldots, m$ and put

$$t_0^i = \inf\{t \in J:\ D_t \cap C_i \neq \emptyset\}$$

and

$$t_1^i = \sup\{t \in J\colon\ E_t \cap C_i \neq \emptyset\}.$$

Consider three cases.

(a) $t_1^i < t_0^i$. Then there is no $t \in J$ such that $t_1^i < t < t_0^i$ (otherwise, $(D_t \cup E_t) \cap C_i = \emptyset$, but $C_i \cap \operatorname{cl} U \neq \emptyset$ and $\operatorname{cl} U_i = D_t \cup E_t$, a contradiction). Put $T_i = [t_1^i, t_0^i]$. According to Theorem 2 in [4], p. 215, the set $K_i = \operatorname{cl}(D_{t_0^i} \setminus D_{t_1^i})$ is an indecomposable continuum. Moreover, it has the nonempty interior in X by (21.1). Hence, by (19), there is a point $a' \in K_i$ such that $K_i \subset L$ provided

$$a' \in L \in C(X) \quad \text{and} \quad L \cap (X \setminus K_i) \neq \emptyset.$$

Put $A_i = \{a'\}$.

(b) $t_0^i = t_1^i$. Then we put $T_i = \{t_0^i\}$, $A_i = \emptyset$.

(c) $t_0^i < t_1^i$. Put $T_i = [t_0^i, t_1^i]$. The set $T_i \cap J$ is finite. In fact, suppose, on the contrary, that $T_i \cap J$ is infinite. According to the symmetric properties of $\mathscr{D}$ and $\mathscr{E}$ in K, we may assume that there are $t_k \in J$ such that $t_0^i < t_k < t_{k+1} < t_1^i$ for $k = 1, 2, \ldots$ Let

$$P = D_{t_0^i} \cup C_i \cup \bigcap_{k=1}^{\infty} E_{t_k}.$$

Then P is a continuum. Write

$$P_k = D_{t_k} \cup P \cup \bigcup \{C_j\colon\ C_j \cap (P \cup D_{t_k}) \neq \emptyset\} \quad \text{for } k = 1, 2, \ldots$$

Since P_k is a countable monotonic collection of proper (by (21.1)) subcontinua of X and

$$X = \bigcup_{k=1}^{\infty} P_k,$$

we obtain a contradiction, because $X \in \mathscr{F}$.

Since $T_i \cap J$ is finite, we find $t_1, \ldots, t_n \in J$ such that

$$t_0^i = t_1 < t_2 < \ldots < t_n = t_1^i$$

and $D_{t_{k+1}}$ and D_{t_k} form a jump in $\mathscr{D}$. Therefore, by Theorem 2 in [4], p. 215, the sets $\operatorname{cl}(D_{t_{k+1}} \setminus D_{t_k})$ are indecomposable continua. From (19), similarly as in the case (a), we find a finite set A_i such that $K_i \subset L$ provided

$$A_i \subset L \in C(X) \quad \text{and} \quad L \cap (X \setminus K_i) \neq \emptyset,$$

where $K_i = \operatorname{cl}(D_{t_1^i} \setminus D_{t_0^i})$.

Let $(s_1^0, s_1^1), (s_2^0, s_2^1), \ldots, (s_n^0, s_n^1)$ be the collection of disjoint open intervals in $[0, 1]$ the union of which is a complement of $\bigcup_{i=1}^{m} T_i$. Then

$$K_j' = \operatorname{cl}(D_{s_j^1} \setminus D_{s_j^0})$$

is a continuum irreducible between

$$L_j = D_{s_j^0} \cup \bigcup \{C_i\colon\ C_i \cap D_{s_j^0} \neq \emptyset\}$$

and

$$R_j = E_{s_j^1} \cup \bigcup \{C_i\colon\ C_i \cap E_{s_j^1} \neq \emptyset\}.$$

Take an arbitrary point $b_j \in K_j' \cap L_j$ and an arbitrary point $c_j \in K_j' \cap R_j$. Fix $c \in X \setminus K$. The set

$$A = \{c\} \cup \bigcup_{i=1}^{m} A_i \cup \bigcup_{j=1}^{n} \{b_j, c_j\}$$

is a set for which we were looking for. The proof of (21) is complete.

Now we have

(22) *If* $X \in \mathscr{F}$ *and* K *is an irreducible continuum in* X, *then there is a finite set* $A \subset X$ *such that* $\operatorname{int} K \subset M$ *provided* $A \subset M \in C(X)$.

In fact, it follows from (6) and (9) that $\operatorname{int} K$ has a finite number of components $U_1, U_2, \ldots, U_n$. The sets $\operatorname{cl} U_1, \operatorname{cl} U_2, \ldots, \operatorname{cl} U_n$ are irreducible continua by (20). Therefore, there are finite sets $A_i \subset X$ such that $U_i \subset M$ provided $A_i \subset M \in C(X)$ for $i = 1, 2, \ldots, n$ by (21). The set A defined by the equality

$$A = A_1 \cup A_2 \cup \ldots \cup A_n$$

has the required properties.

(23) *If*

$$\{a_1, a_2, \ldots, a_n\} \subset X \in \mathscr{F}, \quad \{R_0, R_1, \ldots, R_n\} \subset C(X),$$

$$X = R_0 \cup R_1 \cup \ldots \cup R_n$$

and R_i *is irreducible between* a_i *and* $R_0 \cup R_1 \cup \ldots \cup R_{i-1}$ *for* $i = 1, 2, \ldots, n$, *then there is a finite set* $A \subset X$ *such that* $X \setminus R_0 \subset M$ *provided* $A \subset M \in C(X)$.

We proceed by induction. For $n = 1$ Proposition (23) is an immediate consequence of (22) (cf. (2)). Assume now that (23) is true for n. Let

$$\{b_1, b_2, \ldots, b_{n+1}\} \subset X \in \mathscr{F}, \quad \{K_0, K_1, \ldots, K_{n+1}\} \subset C(X),$$

$$X = K_0 \cup K_1 \cup \ldots \cup K_{n+1}$$

and K_i is irreducible between b_i and $K_0 \cup K_1 \cup \ldots \cup K_{i-1}$ for $i = 1, 2, \ldots, n+1$. Put $a_i = b_{i+1}$, $R_0 = K_0 \cup K_1$, $R_i = K_{i+1}$ for $i = 1, 2, \ldots, n$. According to the assumption there is a finite set A such that

$$X \setminus R_0 = X \setminus (K_0 \cup K_1) \subset M$$

provided $A \subset M \in C(X)$. Then

$$\mathrm{cl}(X \setminus (K_0 \cup K_1)) \subset M$$

provided $A \subset M \in C(X)$. According to (22) there is a finite set B such that $\operatorname{int} K_1 \subset M$ provided $B \subset M \in C(X)$. Let

$$A \cup B \subset M \in C(X).$$

Then $\operatorname{int} K_1 \cup \mathrm{cl}(X \setminus (K_0 \cup K_1)) \subset M$. Since

$$(X \setminus K_0) \cap (X \setminus \mathrm{cl}(X \setminus (K_0 \cup K_1))) \subset \operatorname{int} K_1,$$

we conclude that the set $A \cup B$ has the required property.

J. B. Fugate has asked the following: Suppose $X \in \mathscr{F}$. Is X irreducible about some finite set? (University of Houston Mathematics Problem Book, Problem 113). The following theorem gives the answer:

(24) THEOREM. *If $X \in \mathscr{F}$, then X is finitely irreducible.*

Proof. Consider the following collection $\mathscr{P} \subset C(X)$:

$\mathscr{P} = \{K \in C(X)$: there exists a finite set A such that $A \cup K \in S(X)\}$.

First we prove that

(24.1) *if $K_n \in \mathscr{P}$, $K_{n+1} \subset K_n$ for $n = 1, 2, \ldots$, then*

$$K = \bigcap_{n=1}^{\infty} K_n \in \mathscr{P}.$$

In fact, let A_n be a finite set such that $A_n \cup K_n \in S(X)$. We can assume that $a_1, a_2, \ldots$ is a sequence of points of X such that for each $n = 1, 2, \ldots$ there is a positive integer k_n such that

$$A_n \subset \{a_1, a_2, \ldots, a_{k_n}\}.$$

By induction we construct a sequence $\{R_i\}$ of subcontinua of X such that $R_0 = K$ and R_{n+1} is an arbitrary continuum in X irreducible between a_{n+1} and $R_0 \cup \ldots \cup R_n$. Put

$$Q_n = R_0 \cup \ldots \cup R_n.$$

Then $Q_n \subset Q_{n+1}$ and Q_n are continua for $n = 1, 2, \ldots$ Observe that

$$X = \bigcup_{n=1}^{\infty} Q_n.$$

Indeed, if $x \in X \setminus K$, then there is a positive integer n such that $x \in X \setminus K_n$. The continuum Q_{k_n} contains all points of the set A_n and intersects K_n; thus $K_n \cup Q_{k_n} = X$ because $A_n \cup K_n \in S(X)$. Hence $x \in Q_{k_n}$.

The relations

$$\bigcup_{n=1}^{\infty} Q_n = X \in \mathscr{F}$$

imply that $Q_n = X$ for some positive integer n. Applying (23) we find a finite set $A \subset X$ such that $X \setminus K \subset M$ provided $A \subset M \in C(X)$. This means that $A \cup K \in S(X)$, i.e., $K \in \mathscr{P}$.

By Brouwer's reduction theorem and (24.1) we infer that

(24.2) *there exists a minimal element of* $\mathscr{P}$.

Fix an arbitrary minimal element K of $\mathscr{P}$ (cf. (24.2)). There exists a finite set A such that $A \cup K \in S(X)$. Then $K \in I(A, X)$. Moreover, we conclude that $K \in I_0(A, X)$ by (5) and the minimality of K in $\mathscr{P}$. It follows from (11) that $K \in \mathscr{F}$. From (16) it follows that if K is nondegenerate, then there are a point a and a proper subcontinuum N of K such that $\{a\} \cup N \in S(K)$. Proposition (22) implies that there is a finite set B such that $\operatorname{int}(\operatorname{cl}(K \setminus N)) \subset M$ provided that $B \subset M \in C(X)$. We prove that

$$A \cup B \cup N \in S(X).$$

Let $Q \in C(X)$ be such that $A \cup B \cup N \subset Q$. Note that $Q \cup K = X$ because $A \cup K \subset Q \cup K$ and $Q \cup K \in C(X)$. Hence

$$X \setminus Q \subset K \setminus N \subset \operatorname{cl}(K \setminus N).$$

Since $X \setminus Q$ is open, $X \setminus Q \subset \operatorname{int}(\operatorname{cl}(K \setminus N))$. Moreover, $B \subset Q \in C(X)$, and so

$$\operatorname{int}(\operatorname{cl}(K \setminus N)) \subset Q.$$

Therefore $Q = X$.

Thus $A \cup B \cup N \in S(X)$, i.e., $N \in \mathscr{P}$, contrary to the choice of K. The proof is complete.

Proposition (8) and Theorem (24) imply

(25) COROLLARY. *A metric continuum X is irreducible about a finite set if and only if X is not the union of a countable monotonic collection of proper subcontinua.*

Properties of $\mathscr{F}$ (almost all propositions proved here) used in the proof of Theorem (24) imply also some new information about the properties of finitely irreducible continua.

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INSTITUTE OF MATHEMATICS
UNIVERSITY OF WROCŁAW

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