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ON THE NOTION OF VIRTUAL AMENABILITY FOR GROUPS

BY

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The concept of amenability for locally compact groups is generalized, à la Mackey–Ramsay analysis, to a notion of virtual amenability. Definitions, examples, and theorems are given showing the differences and interplay between this generalization and other notions of amenability. Finally, virtual amenability is shown to be related to Margulis' property T for pairs (G, N).

Introduction. Since the original definition of amenability for abstract groups was given (a group G is *amenable* if there exists a left-invariant mean on the Banach space of all bounded functions on G), a variety of sophisticated generalizations has been introduced. There is, for example, amenability for locally compact groups, for Banach algebras, for group actions, for groupoids, etc. In this paper we discuss a different notion of amenability for a locally compact group G, a notion intimately connected with the representation theory of G and the way closed, normal, amenable subgroups are imbedded in G. It was, in fact, our discovery of some nonamenable groups that nevertheless display in their representation theory some properties reminiscent of the very definition of amenability, that motivated this article.

The formulation of amenability for locally compact groups from which we begin is that the trivial representation be weakly contained in the regular representation. If we regard the regular representation as a representation induced from a normal subgroup (the trivial subgroup), then a natural question arises as to whether there are nonamenable groups G for which there exists some normal subgroup N such that the trivial representation of G is weakly contained in the set of all representations of g that are induced from N. This notion is somewhat naive, as we shall now indicate. If we take N to be G, then this condition always holds, so that it seems appropriate to require that the normal subgroup itself be proper.

And we notice that if this condition holds for some amenable N, then the trivial representation of G is weakly contained in the representation of G that is induced from the regular representation of N, i.e., in the regular representation of G. This is so since, for an amenable group N, the regular representation weakly contains every representation. Hence, if the condition holds for an amenable N, then G is itself amenable. Obviously, the converse holds, so that this generalization leads nowhere when N is amenable. In Section 1, we study this a bit further, allowing N to be nonamenable.

Even if N is not amenable, we note that the trivial representation of G is weakly contained in the representation of G induced from the trivial representation of N if and only if G/N is amenable. It seems appropriate, in order to avoid such a trivial reduction to ordinary amenability, to hypothesize that the trivial representation of G be weakly contained in the set of all representations of G induced from nontrivial irreducible representations L of N. However, if $G = K \times K$ for K a compact group, and if N is one of the factor subgroups, then we see that the trivial representation of G is not weakly contained in the set of all representations induced from nontrivial irreducible representations of N, even though G is itself amenable. This strengthening of our proposed condition also seems inconsistent.

Recalling that the trivial representation of a compact group is always contained in the tensor product $T \otimes \bar{T}$ of any representation T with its conjugate representation $\bar{T}$, we are led to the following definition, the one we shall discuss throughout the rest of this paper.

DEFINITION. Let N be a closed normal subgroup of a locally compact group G. We say that (G, N) is an *amenable pair* if the trivial representation of G is weakly contained in the set of all tensor product representations $U^L \otimes \bar{U}^L$, where L runs over the set of nontrivial irreducible representations of N.

We will show that amenable pairs exist even when N or G/N is not amenable. In Section 2, we generalize this definition slightly, by allowing representations of G which are "virtually induced" from N, i.e., from the action of G on the dual space $\hat{N}$ of N (see Section 2 for the precise definitions). We define what we call "virtually amenable pairs" and demonstrate the existence of a proper class of "virtually amenable" groups containing the amenable ones as a proper subclass. Finally, in Section 3, we establish a relationship between this notion of virtual amenability and Margulis' definition of pairs (G, N) having property T.

1. Amenable pairs. Let G be a locally compact group and let N be a closed normal subgroup of G. We restate our basic definition.

DEFINITION. The pair (G, N) is called an *amenable pair* if the trivial representation I_G of G is weakly contained in the set $U^L \otimes \bar{U}^L$ of all tensor products of representations U^L of G, which are induced from nontrivial irreducible representations L of N, with their complex conjugate representation.

1.1. PROPOSITION. *The following are equivalent for a locally compact group G:*

(i) *G is amenable.*

(ii) *The pair (G, N) is amenable for every nontrivial, closed, normal, amenable subgroup N of G.*

(iii) *The pair (G, N) is amenable for some closed, normal, amenable subgroup N of G.*

(iv) *The pair (G, G_A) is amenable, where G_A is the maximum closed normal amenable subgroup of G.*

Proof. We have (i)$\Rightarrow$(ii)$\Rightarrow$(iv)$\Rightarrow$(iii) (with the remark that when N is compact, a slightly different argument may be necessary to see that (i)$\Rightarrow$(ii)). To see that (iii)$\Rightarrow$(i), observe first that the set of all nontrivial irreducible representations L of an amenable subgroup N is contained in the spectrum of the regular representation λ_N of N. And, recall that the tensor product of the regular representation λ_G of G with its conjugate is weakly equivalent to λ_G. But then (iii) implies, by using continuity of inducing and continuity with respect to tensor products, that I_G is contained in the spectrum of the regular representation λ_G of G, whence G is amenable.

It is possible for a pair (G, N) to be amenable even though G (and/or N) is not. Indeed, if G is any non-Kazhdan group (property T, see [1]), then (G, G) is an amenable pair. Also, if G is a direct product NH and if N is not a Kazhdan group (amenable or not), then (G, N) is an amenable pair if and only if H is amenable. For, in this direct product case, U^L is just the outer Kronecker product of L with the regular representation of H. The only way tensor products of such representations of NH can weakly contain the trivial representation I_G of G is for I_H to be weakly contained in the regular representation λ_H, i.e., that H be amenable.

The contents of the preceding proposition is that when N is amenable, (G, N) is an amenable pair if and only if G itself is an amenable group. If N is a nonamenable, non-Kazhdan group and G is the direct product NH with H amenable, then (G, N) is an amenable pair even though G is not an amenable group. Much more subtle and interesting examples arise when we generalize to "virtually amenable" pairs.

2. Virtually amenable pairs. Again let G be a locally compact group and let N be a closed normal subgroup of G. We generalize our preceding notion of amenable pairs by extending the definition of an induced representation, à la Mackey and Ramsay ([2] and [4]), to that of a virtually induced one. Let us recall this definition.

Let G be a locally compact group, let (S, μ) be a probability space on which G acts, by $s \to s \cdot g$, as a group of quasi-invariant ergodic transformations, i.e., jointly ergodic measurable transformations of S that preserve the null sets of μ, and let R be a measurable map of $S \times G$ into the unitary group on a Hilbert space K. Then R is called a *virtual representation* or a *μ-cocycle* of $S \times G$ if R satisfies the following "cocycle identity":

$$R(s, gg') = R(s, g) R(s \cdot g, g')$$

for almost all $s \in S$ and $g, g' \in G$.

A virtual representation R is *irreducible* if the only operator-valued functions T on S which *intertwine* R with itself,

$$T(s) R(s, g) = R(s, g) T(s \cdot g),$$

are the constant scalar-valued functions.

Given a cocycle R of $S \times G$, we may define a representation U^R of G, acting in the Hilbert space $L^2(S, \mu, K)$, by

$$[U^R g(f)](s) = \varrho(s, g)^{1/2} R(s, g)(f(s \cdot g)),$$

where $\varrho(s, g)$ is the Radon–Nikodym derivative of the measure μ with respect to its transform under the action of g. The function ϱ exists because the transformations of S defined by elements of G are all quasi-invariant.

Since R satisfies the cocycle identity above, this formula for U^R defines a unitary representation of G. We say that U^R is *virtually induced* from the virtual representation R.

Now, if N is a closed normal subgroup of a locally compact group G, then G acts by inner automorphisms on N and, consequently, acts on the dual space $\hat{N}$ of N: $(L \cdot g)_n = L_{gng^{-1}}$. If N is of type I, this action on $\hat{N}$ is Borel. For any quasi-invariant probability measure μ on $\hat{N}$ and any μ-cocycle R, we may form the virtually induced representation U^R. Since the action of N on $\hat{N}$ is trivial, we see from the cocycle identity that any cocycle R of $\hat{N} \times G$ must in fact satisfy

$$R(L, nn') = R(L, n) R(L, n'),$$

hence defines an ordinary representation of N for each point L. It is natural in this $\hat{N}$ context to require that all the cocycles satisfy $R(L, n) = L_n$ for all $L \in \hat{N}$ and $n \in N$. Ordinary induced representations from N are included in these more general ones. Indeed, let us write η for the projection of G onto G/N, and let γ denote a Borel cross-section of G/N back into G, i.e., $\pi(\gamma(s)) \equiv s$ for all $s \in G/N$. If $L \in \hat{N}$, we let μ be an ergodic quasi-invariant probability measure on $\hat{N}$ which is concentrated on the G orbit $L \cdot G$ of L. We define a μ-cocycle R of $\hat{N} \times G$ into the unitary group on the Hilbert space of L by $R(s, g) = I$ (the identity operator) if

$$s \notin L \cdot G \quad \text{and} \quad R(L \cdot g', g) = R\big(L \cdot g', n\gamma(\pi(g))\big) = L_{\gamma(\pi(g'))n\gamma(\pi(g))[\gamma(L\pi(g'g))]^{-1}}.$$

One checks directly that this definition of R satisfies the cocycle identity. The resulting virtually induced representation U^R coincides with one of the usual formulas for the ordinary induced representation U^L (see [4]).

With these definitions made, the following seems natural:

Definition. Let N be a closed normal subgroup of a locally compact group G. The pair (G, N) is called *virtually amenable* if I_G is weakly contained in the set $U^R \otimes \bar{U}^R$ of tensor products of representations U^R of G, which are virtually induced from nontrivial irreducible virtual representations R of the $\hat{N} \times G$, with their complex conjugate representation.

A μ-cocycle R is called *trivial* if μ is the point mass δ_{1_N}.

By analogy with the equivalences in Proposition 1.1, we give the following

DEFINITION. A locally compact group G is *virtually amenable* if the pair (G, G_A) is virtually amenable, where G_A is the maximum closed normal amenable subgroup of G.

We have the following perhaps expected result:

2.1. PROPOSITION. *The following are equivalent for a locally compact group G:*

(i) *G is virtually amenable.*

(ii) *The pair (G, N) is virtually amenable for every nontrivial, closed, normal, amenable subgroup N of G.*

(iii) *The pair (G, N) is virtually amenable for some closed, normal, amenable subgroup N of G.*

We omit the proof, except to say that the only nontrivial part ((i) $\Rightarrow$ (ii)) follows from the inducing in stages result for virtual induction (see [4]).

An immediate consequence of Proposition 2.1 is

2.2. PROPOSITION. *If G is a locally compact group, K is a closed normal amenable subgroup, and G/K is virtually amenable, then G is virtually amenable.*

Interestingly, the converse to Proposition 2.2 is not true. Indeed, since the pair $(G, \{e\})$ is never virtually amenable (there are no nontrivial irreducible representations of $\{e\}$), any group G for which $G_A = e$ (what might be called a *totally nonamenable group*) is not a virtually amenable group. To find a counterexample to the converse of Proposition 2.2, it is only necessary to find a nonamenable, but virtually amenable, group G. For then G/G_A would not be virtually amenable (see below).

We remark that an amenable pair (G, N) is certainly virtually amenable, whence (G, G) is virtually amenable for any G which is not a Kazhdan group.

As a first example, let F be a discrete free group on two generators, algebraically thought of as a subgroup of SO(3), and let G be the semidirect product $\boldsymbol{R}^3 * F$. Then G is not amenable. Therefore, by Proposition 1.1, the pair $(G, \boldsymbol{R}^3)$ is not amenable. But

2.3. PROPOSITION. *The pair $(G, \boldsymbol{R}^3)$ is virtually amenable.*

Proof. Take μ_j to be normalized Lebesgue measure on the sphere of radius $1/j$ in $\hat{\boldsymbol{R}}^3$ and define the μ_j-cocycle $R_j(\chi, g) = R_j(\chi, x\sigma)$ to equal $\chi(x)$

(for $x \in \boldsymbol{R}^3$, $\sigma \in F$, and $\chi \in \hat{\boldsymbol{R}}^3$). Let f_j be the constant function 1 in $L^2(\hat{\boldsymbol{R}}^3, \mu, \boldsymbol{C})$. Then

$$([U_{x\sigma}^{(R_j)}(f_j)], f_j) = \int [U_{x\sigma}^{(R_j)}(f_j)](\chi)\, d\mu_j(\chi) = \int \chi(x)\, d\mu_j(\chi),$$

and this clearly tends to 1 uniformly on compacta in G. It is enough to show that the trivial representation I_G is weakly contained in the set of all nontrivial virtually induced representations U^R from $\boldsymbol{R}^3$, and this implies that $(G, \boldsymbol{R}^3)$ is virtually amenable

Sullivan [5] has proved the existence of a subgroup Γ of SO(5) which has property T, i.e., is a Kazhdan group. Obviously, a construction similar to that given above would produce a virtually amenable group having a noncompact Kazhdan quotient. Of course, it is impossible for a noncompact Kazhdan group itself to be virtually amenable.

The example of Proposition 2.3 is a semidirect product $N * H$, where the normal subgroup $N(\boldsymbol{R}^3)$ is not regularly imbedded in the sense of Mackey (see [2]). The stability subgroups for the nontrivial irreducible representations L of N in this example are, however, all amenable. We have already seen in Section 1 that if G is a direct product NH, with N not a Kazhdan group and H amenable, then the pair (G, N) is amenable, whence virtually amenable. In such a direct product, N is always regularly imbedded, but the stability subgroups are always G itself, hence as nonamenable as is G. The notion of virtual induction was invented precisely to handle non-regularly imbedded normal subgroups, so the following lemma may shed some light.

2.4. Lemma. *Suppose G is a locally compact group and that N is a closed normal subgroup of G for which*

(i) *N is of type I and is regularly imbedded in G;*

(ii) *each Mackey little group H_L/N for a nontrivial irreducible representation L of N is amenable.*

Then (G, N) is an amenable pair if and only if (G, N) is a virtually amenable pair.

Proof. Hypotheses (i) and (ii) are precisely what is needed (see [2]) to show that each virtually induced representation U^R of G is weakly contained in an ordinary induced representation U^L. In that case, virtual amenability and amenability are identical.

As a final example, let G be the semidirect product $\boldsymbol{R}^2 * \mathrm{SL}(2, \boldsymbol{R})$ with respect to the usual linear action. Then G is not amenable. Again, by 1.1, $(G, \boldsymbol{R}^2)$ is not an amenable pair. This time $(G, \boldsymbol{R}^2)$ is also not a virtually amenable pair. Indeed, $\boldsymbol{R}^2$ is regularly imbedded in G and each stability subgroup H_χ, for χ nontrivial in $\hat{\boldsymbol{R}}^2$, is isomorphic to a semidirect product of the form $\boldsymbol{R}^2 * \boldsymbol{R}$. Since these stabilizers are all amenable, the preceding lemma would imply that $(G, \boldsymbol{R}^2)$ is amenable if it is virtually amenable.

Remark. One might expect there to be some connection between the notion of a virtually amenable pair (G, N) and the concept of amenability for the groupoid $\hat{N} \times G$ (see [6]). In connection with this, we observe that in the preceding example the ergodic groupoids $(\hat{\boldsymbol{R}}^2 \times \mathrm{SL}(2, \boldsymbol{R}), \mu)$ are all amenable groupoids whenever μ is not the point mass δ_0 (the action here is transitive so that the groupoid is similar to its amenable stabilizer), and yet the pair $(G, \boldsymbol{R}^2)$ is not virtually amenable. Hence the notion of virtual amenability seems to be distinct from that of amenable group actions.

3. Margulis pairs. Let G and N be as in the preceding sections.

DEFINITION. The pair (G, N) is called a *Margulis pair* if, whenever $I_G = \lim \pi_j$ for $\{\pi_j\}$ a sequence of irreducible representations of G, for j large enough π_j is trivial on N.

Margulis introduced this concept in [3], and his terminology was that the pair (G, N) "satisfied a property T".

This definition is clearly related to Kazhdan's property. Indeed, it is clear that G is a Kazhdan group if and only if the pair (G, G) is a Margulis pair. Equally immediate is that G is a Kazhdan group if and only if there exists an N such that G/N is a Kazhdan group and (G, N) is a Margulis pair.

As Margulis himself shows in [3], the pair $(G, \boldsymbol{R}^2)$ is a Margulis pair if

$$G = \boldsymbol{R}^2 * \mathrm{SL}(2, \boldsymbol{R}).$$

This is the same pair which we have seen in Section 2 is not virtually amenable. The general situation is this:

3.1. THEOREM. *Let G be a locally compact group and let N be a closed normal subgroup of G which is of type I. Then the pair (G, N) is virtually amenable if and only if it fails to be a Margulis pair.*

Proof. Suppose (G, N) is virtually amenable. Then I_G is weakly contained in the set of all representations of G of the form $U^R \otimes \bar{U}^{\bar{R}}$ for R running through the nontrivial irreducible cocycles of $\hat{N} \times G$. It is clear from the formula for U^R that there is no subrepresentation of U^R which is trivial on N, and therefore there is no subrepresentation of $U^R \otimes \bar{U}^{\bar{R}}$ which is trivial on N. It must be then that $I_G = \lim \pi_j$, where each π_j is irreducible and not trivial on N, i.e., (G, N) is not a Margulis pair.

Conversely, if (G, N) is not a Margulis pair, let $I_G = \lim \pi_j$. According to Mackey-Ramsay theory (see [2] and [4]), each irreducible π_j is virtually induced from some cocycle R_j, i.e., $\pi_j \equiv U^{R_j}$. Since π_j is not trivial on N, it must be that R_j is a nontrivial cocycle. Also, it is clear that I_G is weakly contained in the set of all $U^{R_j} \otimes \bar{U}^{\bar{R}_j}$, which proves that (G, N) is virtually amenable.

Finally, therefore, we have:

3.2. THEOREM. *A locally compact group G is virtually amenable if and* *only if for each nontrivial, closed, normal, amenable subgroup N of G there exists a sequence* $\{\pi_j\}$ *of irreducible representations of G such that*

(i) π_j *converges to* I_G;

(ii) *the restriction of* π_j *to N is not trivial for any j.*

Proof. Indeed, these two conditions are precisely the statement that (G, N) is not a Margulis pair.

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