COLLOQUIUM MATHEMATICUM

VOL. XLI 1979 FASC. 1

ON HALLIAN DIGRAPHS, PERMANENTS AND TRANSVERSALS

BY

MIROSŁAW ARCZYŃSKI, MIECZYSŁAW BOROWIECKI (ZIELONA GÓRA) AND MACIEJ M. SYSŁO (WROCŁAW)

1. Introduction. Throughout the paper, $G = (X, U)$ denotes a *digraph*, i.e., a finite directed graph without multiple arcs (loops are admitted), where X is the set of vertices and U is the set of arcs.

A *circuit* of G is a sequence of different vertices $x_1, x_2, \ldots, x_p$ such that (x_i, x_{i+1}) for $i = 1, 2, \ldots, p-1$ and (x_p, x_1) are arcs of G. A digraph $G' = (X, V)$, where $V \subset U$, is called a *partial digraph* of G. If each vertex of G is of outdegree and indegree 1, then G is a *permutation digraph*. A digraph G is *hallian* if G contains a partial permutation digraph.

It is easy to see that G is hallian if and only if there exists a system of pairwise disjoint circuits which cover all vertices of G.

Let $G = (X, U)$ be a digraph and let

$$\Gamma_G(x) = \{y\colon (x, y) \in U\} \quad \text{for } x \in X,$$

$$\Gamma_G(A) = \bigcup_{x \in A} \Gamma_G(x) \quad \text{for } A \subseteq X.$$

The following proposition is merely a reformulation of the famous theorem of Ph. Hall (see [2], Theorem 5.1.2).

PROPOSITION 1.1. *A digraph $G = (X, U)$ is hallian if and only if*

$$|\Gamma_G(A)| \geqslant |A|$$

for every non-empty set $A \subseteq X$.

A digraph $G = (X, U)$ is said to have a *hypoproperty* $\mathcal{P}$ if G does not have property $\mathcal{P}$ but $G-x$ has property $\mathcal{P}$ for every vertex $x \in X$, where $G-x$ denotes the subdigraph generated by the set $X - \{x\}$.

The main purpose of this paper is to show that there exists no *hypohallian* digraph. We shall prove also that if a digraph $G = (X, U)$ with n vertices has at least $\{n/2\}+1$ hallian subdigraphs of the form $G-x$ ($x \in X$), then G is hallian.

Let us define now hallian digraphs in terms of permanents and transversals.

Let $A = (a_{ij})$ be a 0-1 square matrix of dimension n. There exists a one-to-one correspondence between the family of 0-1 matrices and the family of digraphs. Let $G(A)$ denote the digraph associated with A. The vertices of $G(A)$ correspond to rows and columns of A, and (i, j) is an arc of $G(A)$ if and only if $a_{ij} \neq 0$. If G is a digraph, then the adjacency matrix of G is its corresponding 0-1 matrix.

One can easily prove now the following

PROPOSITION 1.2. *A digraph $G(A)$ is hallian if and only if* $\operatorname{per}(A) \neq 0$.

Let $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ be a system of subsets of $N = \{1, 2, \ldots, n\}$. There exists also a one-to-one correspondence between the family of such systems and the family of digraphs. Namely, let $G(S)$ denote the digraph associated with $\mathcal{S}$. The vertices of $G(S)$ correspond to elements of N and $\Gamma_{G(S)}(i) = S_i$ $(i \in N)$. If $G = (X, U)$ is a digraph, then $\mathcal{S} = \{\Gamma_G(x): x \in X\}$ is its corresponding system of subsets.

We shall say that the family $\mathcal{S} = \{S_i: i \in N\}$ satisfies *Hall's condition* if the inequality

$$\Big|\bigcup_{i \in M} S_i\Big| \geqslant |M|$$

is valid for every finite subset M of N. We shall, for brevity, write

$$S(M) = \bigcup_{i \in M} S_i.$$

As a consequence of the above we have

PROPOSITION 1.3. *A digraph $G(S)$ is hallian if and only if the family $\mathcal{S}$ has a transversal, i.e., if and only if $\mathcal{S}$ satisfies Hall's condition.*

The next section contains the main results of this paper presented in terms of transversals and the equivalent formulations of the results but in terms of digraphs and permanents are contained in Section 3.

The reader is referred to [1]-[3] for other terms not defined here.

2. Main results. Let $\mathcal{S} = \{S_i: i \in N\}$ be a system of subsets of N and let us define systems

$$\mathcal{S}_l = \{S_1^l, S_2^l, \ldots, S_{l-1}^l, S_{l+1}^l, \ldots, S_n^l\} \quad (l \in N),$$

where $S_i^l = S_i - \{l\}$ $(i \in N)$. It is clear that the digraph $G(S) - l$ corresponds to the system $\mathcal{S}_l$ for every $l \in N$. We prove now the main theorem of this paper.

THEOREM 2.1. *If a family $\mathcal{S} = \{S_i: i \in N\}$ of subsets of N has no transversal, i.e., if there exists an $M \subseteq N$ such that $|S(M)| < |M|$, then no $\mathcal{S}_l$ has a transversal for every $l \in S(M)$.*

Proof. Suppose that there exists an $M \subset N$ such that $|S(M)| < |M|$ and let $l \in S(M)$. If $l \in M$, then write $M' = M - \{l\}$. In this case we have

$$|S^l(M')| = \Big|\bigcup_{i \in M'} S_i^l\Big| \leqslant \Big|\bigcup_{i \in M} S_i\Big| - 1 = |S(M)| - 1,$$

since $l \notin S^l(M')$. Therefore

$$|S^l(M')| \leqslant |S(M)| - 1 < |M| - 1 = |M'|.$$

On the other hand, if $l \notin M$, then also

$$|S^l(M)| \leqslant |S(M)| < |M|.$$

Thus, for every $l \in S(M)$, $\mathscr{S}_l$ does not satisfy Hall's condition so that $\mathscr{S}_l$ has no transversal.

Let $\mathscr{T}$ denote the following property: $\mathscr{S}$ has a transversal. If we define $\mathscr{S} - l$ by $\mathscr{S}_l$, then Theorem 2.1 implies

COROLLARY 2.1. *$\mathscr{T}$ is not a hypoproperty of $\mathscr{S}$.*

Let $t(n)$ be the smallest integer such that every system $\mathscr{S} = \{S_i : i \in N\}$ of subsets of $N = \{1, 2, \ldots, n\}$, which has at least $t(n)$ subsystems $\mathscr{S}_l$ having transversals, has also a transversal.

The next theorem computes $t(n)$.

THEOREM 2.2. $t(n) = \{n/2\} + 1$.

Proof. Let us put

$$L = \{l : \mathscr{S}_l \text{ has a transversal}\}.$$

We have to prove now that if $|L| \geqslant \{n/2\} + 1$, then $\mathscr{S}$ has a transversal. Let M be any subset of N and assume first that $L \nsubseteq M$. Hence, there exists an $l \in L - M$, and then $|S^l(M)| \geqslant |M|$, since $\mathscr{S}_l$ satisfies Hall's condition. Therefore $|S(M)| \geqslant |S^l(M)| \geqslant |M|$. Let us consider the opposite case, i.e., $L \subseteq M$. For every $l \in L$ we have $|S^l(M - \{l\})| \geqslant |M| - 1$, and if there exists an $l \in L$ such that $|S^l(M - \{l\})| > |M| - 1$, then

$$|S(M)| \geqslant |S^l(M - \{l\})| \geqslant |M|.$$

Therefore, assume that

$$|S^l(M - \{l\})| = |M| - 1 \quad \text{for every } l \in L. \tag{1}$$

We show now that also in this case $|S(M)| \geqslant |M|$. Indeed, if this inequality does not hold, then $|S(M)| = |M| - 1$ by (1). This implies that there exists an $M_1 \subset N$ such that $|M_1| = |M| - 1$, $S(M) = M_1$, and $S^l(M - \{l\}) = M_1$ for every $l \in L$, since $S_i^l \subseteq S_i$ and $|S^l(M - \{l\})| = |S(M)| = |M| - 1 = |M_1|$. Thus, $l \notin M_1$ for every $l \in L$ so that $L \cap M_1 = \emptyset$. We have $|M_1| = |M| - 1 \geqslant |L| - 1$, since $L \subseteq M$, and by assumption we obtain

$$|M_1 \cup L| = |M_1| + |L| \geqslant |L| - 1 + |L| \geqslant \{n/2\} + \{n/2\} + 1 \geqslant n + 1.$$

We arrive then at a contradiction, since $L, M_1 \subseteq N$, $L \cap M_1 = \emptyset$ and $|N| = n$. Hence $|S(M)| \geqslant |M|$ for every $M \subseteq N$.

Finally, we have to show that this bound cannot be improved. Let n be an odd number and let

$$S_1 = \{2\}, \quad S_n = \{n-1\} \quad \text{and} \quad S_i = \{i-1, i+1\} \quad (i = 2, 3, \ldots, n-1).$$

It is easy to see that L consists of all odd numbers l $(1 \leqslant l \leqslant n)$. Therefore, $|L| = \{n/2\}$ and $\mathscr{S}$ does not have a transversal.

If n is an even number, then let

$$S_1 = \{2\}, \quad S_i = \{i-1, i+1\} \ (i = 2, 3, \ldots, n-2),$$

$$S_{n-1} = \{n-2\}, \quad S_n = \{n\}.$$

In this case again L consists of all odd numbers l $(1 \leqslant l \leqslant n)$. Therefore, $|L| = \{n/2\} = \{(n+1)/2\}$ and $\mathscr{S}$ does not have a transversal.

3. Corollaries. This section presents the main result of this paper in terms of digraphs and permanents.

First, let us consider digraphs.

The following theorem is the counterpart of Theorem 1.1.

THEOREM 3.1. *If a digraph $G = (X, U)$ is not hallian, i.e., if there exists an $A \subseteq X$ such that $|\Gamma_G(A)| < |A|$, then $G - x$ is not hallian for every $x \in \Gamma_G(A)$.*

COROLLARY 3.1. *There exists no hypohallian digraph.*

THEOREM 3.2. *Let $G = (X, U)$ be a digraph with n vertices. If the number of hallian subdigraphs of the form $G - x$ is at least equal to $\{n/2\}+1$, then G is also hallian.*

COROLLARY 3.2. *Every hypohamiltonian digraph is hallian.*

Let now A be a 0-1 square matrix of dimension n and let A_{ii} denote its submatrix obtained by deleting the i-th row and the i-th column $(i = 1, 2, \ldots, n)$.

The results of the preceding section in terms of permanents have the following form:

THEOREM 3.3. *If* $\operatorname{per}(A) = 0$, *then there exists at least one i such that also* $\operatorname{per}(A_{ii}) = 0$.

THEOREM 3.4. *Let A be a* 0-1 *matrix of dimension n. If the number of submatrices A_{ii} which satisfy* $\operatorname{per}(A_{ii}) \neq 0$ *is at least equal to* $\{n/2\}+1$, *then* $\operatorname{per}(A) \neq 0$.

REFERENCES

[1] C. Berge, *Graphs and hypergraphs*, North-Holland and American Elsevier 1973.
[2] M. Hall, Jr., *Combinatorial theory*, Waltham, Mass., 1967.
[3] L. Mirsky, *Transversal theory*, New York 1971.
[4] J. Mitchem, *Hypo-properties in graphs*, p. 223-230 in: G. Chartrand and S. F. Kapoor (editors), *The many facets of graph theory*, Berlin 1969.

INSTITUTE OF MATHEMATICS AND PHYSICS
COLLEGE OF ENGINEERING, ZIELONA GÓRA

INSTITUTE OF INFORMATICS
WROCŁAW UNIVERSITY

Reçu par la Rédaction le 26. 2. 1977;
en version modifiée le 8. 10. 1977