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POWER DOMAINS

BY

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In algebraic topology one uses groups and their homomorphisms in order to study topological properties of spaces and mappings. However, in many cases the transition from topology to algebra by means of the notion of a group is difficult. This situation suggests to use other, more primitive, algebraic structures which may be assigned to topological phenomena less accessible to methods based on the concept of a group. To such notions belongs the notion of a power domain. In the present note I will give some elementary properties of this notion and I will illustrate it by some examples.

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1. Definition and examples. By a *power domain* we understand a system consisting of a set Z and of a family of functions

$$a_m: Z \to Z$$

assigned to indices $m = 0, \pm 1, \pm 2, \ldots$ and satisfying the following conditions:

(1.1) $a_1(z) = z$ for every $z \in Z$,

(1.2) $a_k a_m(z) = a_{k\cdot m}(z)$ for every $z \in Z$ and for $k, m = 0, \pm 1, \pm 2, \ldots$

We denote this power domain by (Z, a_m), or shortly by (Z). If $(Z) = (Z, a_m)$, then $\omega\big((Z)\big)$ denotes the cardinality of the domain (Z), that is the cardinality of the set Z of its elements.

Notice that the power domains are a special kind of the universal algebras (see [2], p. 8).

A power domain (Z, a_m) is said to be *pure* if it satisfies the following condition:

(1.3) *For every $z \in Z$ and every $m = 0, \pm 1, \pm 2, \ldots$ the relation $a_m(z) = a_0(z)$ implies the relation $a_{k+m}(z) = a_k(z)$ for every $k = 0, \pm 1, \pm 2, \ldots$*

Consider some examples illustrating these notions:

(1.4) Example. If Z is a group (with the group operation denoted as multiplication), then, setting $a_m(z) = z^m$ for $z \in Z$ and $m = 0, \pm 1, \pm 2, \ldots$, we get a power domain (Z, a_m). This domain is said to be the *power domain of the group* Z; we denote it by (Z). It is clear that (Z) is pure.

(1.5) Example. Let Z denote the set of all integers. Setting

$$a_m(z) = \begin{cases} 0 & \text{if } m \text{ is even,} \\ m \cdot z & \text{if } m \text{ is odd,} \end{cases}$$

we get a power domain which is not pure, because $a_1 \neq a_3$, although $a_2 = a_0$.

(1.6) Example. Let S^n be the n-dimensional sphere with $n > 0$. Then for every $m = 0, \pm 1, \pm 2, \ldots$ there exists exactly one homotopy class of maps of S^n into itself with a representative γ_m: $S^n \to S^n$ being a map of the Brouwer's degree m. Let X be a space. For every map f: $S^n \to X$ let $[f]$ denote the homotopy class with the representative f. Consider the set Z of all such homotopy classes and let us set

$$a_m([f]) = [f\gamma_m] \quad \text{for every } [f] \in Z.$$

It is clear that a_m: $Z \to Z$ and that both conditions (1.1) and (1.2) are satisfied. The so obtained power domain (Z, a_m) is said to be the *n-th homotopy domain of the space* X; we denote it by $\Delta_n(X)$.

Let us show that $\Delta_n(X)$ is pure. Consider a point $a_0 \in S^n$ and two open and disjoint balls G', G'' (in the space S^n with the spherical metric) contained in $S^n \setminus (a_0)$. One sees easily that, for given integers k and m, there exist two maps γ', γ'': $S^n \to S^n$ such that

$$\gamma' \simeq \gamma_k, \quad \gamma'' \simeq \gamma_m$$

and that

$$\gamma'(S^n \setminus G') = (a_0), \quad \gamma''(S^n \setminus G'') = (a_0).$$

It is well known (see, for instance, [1], p. 44) that the map γ_{k+m} is homotopic to the map $\hat{\gamma}: S^n \to S^n$ (being the join $\gamma' \cdot \gamma''$ of the maps γ', γ'') defined by the formulas:

$$\hat{\gamma}(x) = \begin{cases} \gamma'(x) & \text{if } x \in S^n \setminus G'', \\ \gamma''(x) & \text{if } x \in S^n \setminus G'. \end{cases}$$

Assume now that $f: S^n \to X$ is a map and m is an integer such that $a_m([f]) = a_0([f]) = [f\gamma_0]$. Since γ_0 is null-homotopic, we infer that $f\gamma_m: S^n \to X$ is null-homotopic. Consequently, there exists a map g of the $(n+1)$-dimensional ball Q^{n+1}, bounded by S^n, into X such that

$$g(x) = f\gamma_m(x) \quad \text{for every point } x \in S^n.$$

Now let us notice that there exists a homotopy

$$\psi : Q^{n+1}\times\langle 0, 1\rangle \to Q^{n+1}$$

satisfying the conditions:

$$\psi(x, 0) = x \quad \text{for every point } x \in Q^{n+1},$$
$$\psi(x, t) = x \quad \text{for every } (x, t)\in(S^n\setminus G')\times\langle 0, 1\rangle,$$
$$\psi(Q^{n+1}, 1) = S^n\setminus G'.$$

Setting

$$\varphi(x, t) = g\psi(x, t) \quad \text{for every } (x, t)\in S^n\times\langle 0, 1\rangle,$$

we get a homotopy $\varphi : S^n\times\langle 0, 1\rangle \to X$ such that

$$\varphi(x, 0) = f\gamma_m(x) \text{ and } \varphi(x, 1) = f(a_0) \quad \text{for every point } x\in S^n,$$
$$\varphi(x, t) = f(a_0) \quad \text{for every } (x, t)\in(S^n\setminus G')\times\langle 0, 1\rangle.$$

It follows that

$$f\gamma_{k+m} \simeq f\hat{\gamma} = f(\gamma'\cdot\gamma'') = f\gamma'\cdot f\gamma'' \simeq f\gamma'\cdot f(a_0) = f\gamma' \simeq f\gamma_k.$$

Hence $a_{k+m}([f]) = [f\gamma_{k+m}] = [f\gamma_k] = a_k([f])$, that is condition (1.3) is satisfied and consequently the n-th homotopy domain $\Delta_n(X)$ is pure.

(1.7) Example. Keeping the notations S^n, X, γ_m used in example (1.6), let us denote by Z the collection of all homotopy classes $[f]$ of maps $f : X \to S^n$. Setting

$$a_m([f]) = [\gamma_m f] \quad \text{for every } [f]\in Z,$$

we get a function $a_m : Z \to Z$ and it is clear that conditions (1.1) and (1.2) are satisfied. Thus we get a power domain (Z, a_m) which we call the *n-th cohomotopy domain of the space X*; we denote it by $\Delta^n(X)$. Notice that the n-th cohomotopy domain of a space is defined for every space X, in contrast to the n-th cohomotopy group of X, defined only if we subject X to some restrictive conditions.

(1.8) Problem. Does there exist a space X such that the n-th cohomotopy domain $\Delta^n(X)$ is not pure? (**P 784**).

2. Homomorphisms of power domains. Since power domains are a special kind of universal algebras, the notions of a homomorphism and of an isomorphism (onto) maintain their sense. Explicitly, a *homomorphism* of a power domain (Z, a_m) into another power domain (Z', a'_m) is a function $\varphi : Z \to Z'$ such that

$$\varphi a_m(z) = a'_m\varphi(z) \quad \text{for every } z\in Z \text{ and for } m = 0, \pm 1, \pm 2, \ldots$$

Then we write $\varphi : (Z, a_m) \to (Z', a'_m)$.

A homomorphism $\varphi:(Z, a_m) \to (Z', a'_m)$ is an *isomorphism* if φ is one-to-one (onto). It is then clear that the inverse function φ^{-1} is an isomorphism of (Z', a'_m) onto (Z, a_m).

If there exists an isomorphism $\varphi:(Z, a_m) \to (Z', a'_m)$ then power domains (Z, a_m), (Z', a'_m) are said to be *isomorphic* and we write $(Z, a_m) \simeq (Z', a'_m)$. It is clear that if $(Z, a_m) \simeq (Z', a'_m)$ and if (Z, a_m) is pure, then (Z', a'_m) is also pure.

Let us illustrate these notions by the following examples:

(2.1) Example. If φ is a homomorphism of a group Z into another group Z' then it is clear that φ is also a homomorphism of the power domain (Z) of the group Z into the power domain (Z') of the group Z'.

(2.2) Example. Let X, X' be two spaces and let $g: X \to X'$ be a map. Assigning to every homotopy class $[f]$ with a representative $f: S^n \to X$ the homotopy class $[gf]$ with the representative $gf: S^n \to X'$, we get a homomorphism

$$g_{(n)}: \Delta_n(X) \to \Delta_n(X'),$$

called the homomorphism of the n-th homotopy domain *induced by the map* g. It is clear that $g_{(n)}$ is not changed if we replace g by any map $g': X \to X'$ homotopic to g, and that the identity map $i: X \to X$ induces the identity isomorphism $i_{(n)}: \Delta_n(X) \to \Delta_n(X)$. Moreover, it is clear that $g_{(n)}$ depends covariantly on g, that is if $g: X \to X'$ and $g': X' \to X''$ are maps, then homomorphism $(g'g)_{(n)}: \Delta_n(X) \to \Delta_n(X'')$ induced by the map $g'g: X \to X''$ is the composition $g'_{(n)}g_{(n)}$ of homomorphisms $g_{(n)}: \Delta_n(X) \to \Delta_n(X')$ and $g'_{(n)}: \Delta_n(X') \to \Delta_n(X'')$ induced by g and by g'. It follows that

(2.3) *If X, X' are homotopically equivalent spaces, then their n-th homotopy domains $\Delta_n(X)$ and $\Delta_n(X')$ are isomorphic.*

(2.4) Example. If $g: X \to X'$ is a map, then assigning to every homotopy class $[f']$ with a representative $f': X' \to S^n$ the homotopy class $g^{(n)}([f']) = [f'g]$ with the representative $f'g: X \to S^n$, we get a homomorphism

$$g^{(n)}: \Delta^n(Z') \to \Delta^n(Z),$$

called the homomorphism of the n-th cohomotopy domain *induced by the map* g. It is clear that $g^{(n)}$ is not changed if we replace g by any map homotopic to g and that the identity map $i: X \to X$ induces the identity isomorphism $i^{(n)}: \Delta^n(X) \to \Delta^n(X)$. Moreover, one sees readily that $g^{(n)}$ depends contravariantly on g, that is if $g: X \to X'$ and $g': X' \to X''$ are maps, then $(g'g)^{(n)} = g^{(n)}g'^{(n)}$.

It follows that

(2.5) *If X, X' are homotopically equivalent spaces, then their n-th cohomotopy domains $\Delta^n(X)$ and $\Delta^n(X')$ are isomorphic.*

3. Subdomains. Let (Z, a_m) be a power domain and let Z_0 be a subset of Z such that

$$a_m(Z_0) \subset Z_0 \quad \text{for every } m = 0, \pm 1, \pm 2, \ldots$$

Consider the function $a_{0m}: Z_0 \to Z_0$ defined by the formula

$$a_{0m}(z) = a_m(z) \quad \text{for every } z \in Z_0.$$

It is clear that (Z_0, a_{0m}) is a power domain. We say that this power domain is a *subdomain* of the power domain (Z, a_m) and we write $(Z_0, a_{0m}) \subset (Z, a_m)$. For brevity, we shall say in this case that Z_0 is a subdomain of (Z, a_m).

Notice that the notion of a subdomain is a special case of the notion of a subalgebra (see [2], p. 34), restricted to power domains.

It is clear that each subdomain of a pure power domain is pure itself.

(3.1) Example. If (Z, a_m) is a power domain and $z_0 \in Z$, then the subset Z_0 of Z consisting of all elements of the form $a_m(z_0)$, $m = 0, \pm 1, \pm 2, \ldots$, is a subdomain of (Z, a_m). The cardinality of Z_0 (which is either a natural number or $\aleph_0$) is said to be the *order* of the element z_0.

(3.2) Example. It is clear that if $\varphi: (Z, a_m) \to (Z', a'_m)$ is a homomorphism, then setting $a'_{0m}(z') = a'_m(z')$ for every $z' \in \varphi(Z)$ we get an operation $a'_{0m}: \varphi(Z) \to \varphi(Z)$ such that $(\varphi(Z), a'_{0m})$ is a subdomain of (Z', a'_m). We write $(\varphi(Z), a'_{0m}) = \varphi(Z, a_m)$ and say that $(\varphi(Z), a'_{0m})$ is the *image* of (Z, a_m) by the homomorphism φ.

In particular, if q is an integer, then setting $\varphi(z) = a_q(z)$ for every $z \in Z$ we get a homomorphism $\varphi: (Z, a_m) \to (Z, a_m)$. The image of (Z, a_m) by this homomorphism is a subdomain of (Z, a_m) consisting of all elements $z \in Z$ of the form $z = a_q(z')$ with $z' \in Z$. We denote this subdomain by $q \cdot (Z, a_m)$.

Let us observe that the image of a pure power domain by a homomorphism need not be a pure power domain. In fact, let Z denote the set of all integers and Z' — the set consisting of 0 and of all odd integers.

Setting $\varphi(z) = z$ if $z \in Z'$ and $\varphi(z) = 0$ if $z \in Z \setminus Z'$, we get a function φ mapping Z onto Z'. Now let us set

$$a_m(z) = m \cdot z \quad \text{for every } z \in Z \text{ and for } m = 0, \pm 1, \pm 2, \ldots,$$

$$a'_m(z') = \begin{cases} m \cdot z' & \text{if } z' \in Z' \text{ and } m \text{ is odd,} \\ 0 & \text{if } z' \in Z' \text{ and } m \text{ is even.} \end{cases}$$

One sees easily that (Z, a_m) is a pure power domain and that (Z', a'_m) is not a pure power domain (because $a'_2 = a'_0$, but $a'_3 \neq a'_1$). Moreover, the function φ is a homomorphism of (Z, a_m) onto (Z', a'_m), because

$\varphi a_m(z) = \varphi(m\cdot z)$ for every integers m and z, and if $m\cdot z$ is odd, then $\varphi(m\cdot z) = m\cdot z = a'_m\varphi(z)$, and if $m\cdot z$ is even, then $\varphi(m\cdot z) = 0 = a'_m\varphi(z)$.

(3.3) Example. If $\varphi : (Z, a_m) \to (Z', a'_m)$ is a homomorphism and $(Z'_0, a'_{0m}) \subset (Z', a'_m)$, then setting $Z_0 = \varphi^{-1}(Z'_0)$ we get a subdomain (Z_0, a_{0m}) of (Z, a_m) called the *preimage* of the subdomain (Z'_0, a'_{0m}) of (Z', a'_m) and we write $(Z, a_{0m}) = \varphi^{-1}(Z'_0, a'_{0m})$.

4. Simple domains and their joins. By a *simple domain* we understand a power domain (Z, a_m) in which there exists an element z_0 (called the *null-element* of (Z, a_m)) such that $a_0(z) = z_0$ for every $z\in Z$.

Notice that the power domain (Z) of each group Z is simple, but the n-th homotopy domain $\Delta_n(X)$ is simple if and only if the space X is arcwise connected.

Let $(Z) = (Z, a_m)$ be a finite simple domain with the null-element z_0 and let $\mathfrak{N}$ denote the additive group of integers. Consider the set $\hat{Z}$ consisting of pair $(0, z_0)$ and of all pairs (n, z), where $0\neq n\in\mathfrak{N}$ and $z\in Z$. Setting

$$\hat{a}_m(n, z) = \big(m\cdot n,\ a_m(z)\big) \quad \text{for every } (n, z)\in\hat{Z},$$

one gets a simple domain $(\hat{Z})$ with the null-element $(0, z_0)$. Every power domain isomorphic to $(\hat{Z})$ is said to be a *cluster with the support* (Z). Notice that it is determined (up to an isomorphism) by support (Z) and it is pure if and only if its support (Z) is pure.

Consider an arbitrary set Λ of elements λ (called *indices*) and assume that to every $\lambda\in\Lambda$ there is assigned a simple domain $(Z_\lambda, a_{\lambda,m})$. Let $z_{\lambda,0}$ denote the null-element of $(Z_\lambda, a_{\lambda,m})$ and let Z be the set of all pairs (z, λ) with $z\in Z_\lambda$ and $\lambda\in\Lambda$. By the *join* of domains $(Z_\lambda, a_{\lambda,m})$ we understand the simple domain (Z, a_m), where we identify all pairs of the form $(z_{\lambda,0}, \lambda)$ and where the operation a_m is defined by the formula

$$a_m(z, \lambda) = \big(a_{\lambda,m}(z), \lambda\big) \quad \text{for } (z, \lambda)\in Z \text{ and for } m = 0, \pm1, \pm2, \ldots$$

One sees easily that the join of simple domains $(Z_\lambda, a_{\lambda,m})$ is pure if and only if all these domains are pure.

(4.1) Example. Let $\mathfrak{N}^k$, where k is a natural number, denote the group consisting of all systems $(m_1, m_2, \ldots, m_k)$ of integers with the group operation given by the formula

$$(m_1, m_2, \ldots, m_k) + (m'_1, m'_2, \ldots, m'_k) = (m_1+m'_1, m_2+m'_2, \ldots, m_k+m'_k).$$

Let us observe that if $k > 1$, then the power domain $(\mathfrak{N}^k)$ is the join of $\aleph_0$ domains isomorphic with $(\mathfrak{N}^1)$. It follows that $(\mathfrak{N}^k) \simeq (\mathfrak{N}^l)$ if either $k = l = 1$, or if both numbers k, l are greater than 1.

(4.2) Example. If Z is a finite group, then the power domain $(\mathfrak{N}^k\times Z)$ is, in the case $k = 1$, isomorphic to the join of (Z) and of a cluster with the support (Z). If, however, $k > 1$, then the power domain $(\mathfrak{N}^k \times Z)$ is the join of (Z) and of $\aleph_0$ clusters with the support (Z). It follows

(4.3) *If Z is a finite group and k, l are natural numbers, then* $(\mathfrak{N}^k \times Z) \simeq (\mathfrak{N}^l \times Z)$ *if either $k = l$, or if both numbers k, l are greater than* 1.

5. Quotient domains. Let $(Z_0, a_{0,m})$ be a subdomain of a power domain (Z, a_m). Assign to every element $z \in Z$ the set $[z] \subset Z$ defined as follows:

If $z \in Z \setminus Z_0$, then $[z]$ consists of one element z only.

If $z \in Z_0$, then $[z]$ consists of all $z' \in Z_0$ such that $a_0(z) = a_0(z')$.

Notice that $z \in [z]$ for every $z \in Z$ and that different sets $[z]$ are disjoint. Let $\mathring{Z}$ denote the collection of all sets $[z]$ and let us assign to every $m = 0, \pm 1, \pm 2, \ldots$ a function $\mathring{a}_m : \mathring{Z} \to \mathring{Z}$ defined as follows:

$$\mathring{a}_m([z]) = \begin{cases} [a_m(z)] & \text{if } z \in Z \setminus Z_0, \\ [a_0(z)] & \text{if } z \in Z_0. \end{cases}$$

It is clear that

$$\mathring{a}_1([z]) = [z] \quad \text{for} \quad \text{every } [z] \in \mathring{Z}. \tag{5.1}$$

Let us show that

$$\mathring{a}_n \mathring{a}_m([z]) = \mathring{a}_{m \cdot n}([z]) \quad \text{for every } [z] \in \mathring{Z} \text{ and for } m, n = 0, \pm 1, \ldots \tag{5.2}$$

We distinguish the following three cases:

Case 1. $z \in Z_0$.

Then $a_m(z), a_{m \cdot n}(z) \in Z_0$ and, consequently,

$$\mathring{a}_m([z]) = [a_0(z)],$$

$$\mathring{a}_n \mathring{a}_m([z]) = \mathring{a}_n([a_0(z)]) = [a_0 a_0(z)] = [a_0(z)] = \mathring{a}_{m \cdot n}([z]).$$

Case 2. $z \in Z \setminus Z_0$ and $a_m(z) \in Z_0$.

Then $a_{m \cdot n}(z) = a_n a_m(z) \in Z_0$ and

$$\mathring{a}_n \mathring{a}_m([z]) = \mathring{a}_n([a_m(z)]) = [a_0 a_m(z)] = [a_0(z)] = \mathring{a}_{m \cdot n}([z]).$$

Case 3. $z \in Z \setminus Z_0$ and $a_m(z) \in Z \setminus Z_0$.

Then $z \in Z \setminus Z_0$ and

$$\mathring{a}_n \mathring{a}_m([z]) = \mathring{a}_n([a_m(z)]) = [a_n a_m(z)] = [a_{m \cdot n}(z)] = \mathring{a}_{m \cdot n}([z]).$$

Thus relation (5.2) is proved. It follows by (5.1) and (5.2) that $(\mathring{Z}, \mathring{a}_m)$ is a power domain. We call it the *quotient domain* of (Z, a_m) by Z_0 and denote by $(Z, a_m)/Z_0$.

Remark. Notice that quotient domains constitute a special kind of quotient algebras (see [2], p. 35) with the congruence relation $\equiv$ defined as follows:

$z \equiv z'$ means that either $z = z'$ or $z, z' \in Z_0$ and $a_0(z) = a_0(z')$.

(5.3) Example. Let (Z, a_m) be the n-th homotopy domain $\Delta_n(X)$ of a space X. Hence Z is the set of all homotopy classes $[f]$ of maps $f: S^n \to X$ and $a_m([f]) = [f\gamma_m]$ where $\gamma_m: S^n \to S^n$ is a fixed map with degree m. It is clear that the subset Z_0 of Z consisting of all homotopy classes $[f]$ such that $f: S^n \to X$ is homologically trivial (that is f induces trivial homomorphisms of all homology groups of S^n into the corresponding homology groups of X) is a subdomain of Z. The quotient domain $\Delta_n(X)/Z_0$ will be said to be the *reduced n-th homotopy domain of X*; we denote it by $\mathring{\Delta}_n(X)$. It is clear that every map $\varphi: X \to X'$ induces covariantly a homomorphism $\varphi_*: \mathring{\Delta}_n(X) \to \mathring{\Delta}_n(X')$, and we infer that *reduced n-th homotopy domains of two homotopically equivalent spaces are isomorphic.*

(5.4) Example. Let $(Z, a_m) = \Delta^n(X)$ and let Z_0 denote the collection of all homotopy classes $[f]$, where $f: X \to S^n$ is a homologically trivial map. It is clear that Z_0 is a subdomain of Z. The quotient domain $\Delta^n(X)/Z_0$ will be called the *reduced n-th cohomotopy domain of X*. We denote it by $\mathring{\Delta}^n(X)$. It is clear that every map $\varphi: X \to X'$ induces contravariantly a homomorphism $\varphi^*: \mathring{\Delta}^n(X') \to \mathring{\Delta}^n(X)$ and we infer *that reduced n-th cohomotopy domains of two homotopically equivalent spaces are isomorphic.*

6. Power domains of finite abelian groups. It is clear that power domains $(Z), (Z')$ of two isomorphic groups Z, Z' are isomorphic. In general, the converse is not true. However, it is true in the special case of finite abelian groups.

If Z is an abelian group, then we denote the group operation by addition. The functions a_m in the power domain $(Z) = (Z, a_m)$ are then given by the formula

$$a_m(z) = m\cdot z \quad \text{for every } z \in Z \text{ and for } m = 0, \pm 1, \pm 2, \ldots$$

Denote by $q\cdot Z$ (for every integer q) the subgroup of Z consisting of all elements of the form $q\cdot z$. The power domain $q\cdot(Z)$ (as defined in (3.2)) is then the domain of the group $q\cdot Z$, that is

$$q\cdot(Z) = (q\cdot Z).$$

In the sequel we shall denote by $\mathfrak{N}$ the (additive) group of integers and by $\mathfrak{N}_m$ (for every $m = 2, 3, \ldots$) the group of rests modulo m. Notice that for every natural number $m \geqslant 2$ and for every natural number q

(6.1) $\omega\big((\mathfrak{N}_m)\big) = (m, q)\cdot\omega\big(q\cdot(\mathfrak{N}_m)\big)$, *where (m, q) denotes the greatest common divisor of m and of q.*

Let us prove the following

(6.2) Theorem. *If Z, Z' are finite abelian groups, then isomorphism of the power domains $(Z), (Z')$ implies isomorphism of the groups Z, Z'.*

Proof. It is known (see, for instance, [3], p. 308) that every finite abelian group Z is isomorphic with the group $\mathfrak{N}_{m_1}\times\mathfrak{N}_{m_2}\times\ldots\times\mathfrak{N}_{m_k}$, where $m_1, m_2, \ldots, m_k$ are natural numbers $\geqslant 2$ such that

(6.3) $$m_i \mid m_{i+1} \quad \text{for } i = 1, 2, \ldots, k-1.$$

The system of numbers $(m_1, m_2, \ldots, m_k)$ is uniquely determined by the group Z. Thus, in order to prove theorem (6.2), we may assume that

(6.4) $$Z = \mathfrak{N}_{m_1}\times\mathfrak{N}_{m_2}\times\ldots\times\mathfrak{N}_{m_k}$$

and we have only to show that the system of numbers $(m_1, m_2, \ldots, m_k)$ satisfying condition (6.3) is determined by the power domain (Z).

Notice that the number m_k is determined by the power domain (Z), because it is the least natural number q such that $\omega\big(q\cdot(Z)\big) = 1$. Let

(6.5) $$m_k = p_1^{\nu_1}\cdot p_2^{\nu_2}\ldots p_r^{\nu_r},$$

where $p_1, p_2, \ldots, p_r$ are prime numbers, different from one another, and $\nu_1, \nu_2, \ldots, \nu_r$ are natural numbers. It follows by (6.3) and (6.5) that

(6.6) $$m_i = p_1^{\mu_{i,1}}\cdot p_2^{\mu_{i,2}}\ldots p_r^{\mu_{i,r}} \quad \text{for } i = 1, 2, \ldots, k,$$

where $0 \leqslant \mu_{i,j} \leqslant \nu_j$ for $j = 1, 2, \ldots, r$ and $\mu_{i,j} \leqslant \mu_{i+1,j}$ for $i = 1, 2, \ldots$ $\ldots, k-1$ and for $j = 1, 2, \ldots, r$.

It follows by (6.1) that the quotient $\omega\big((Z)\big):\omega\big(p_j\cdot(Z)\big)$ is equal to p_j^k, if $\mu_{i,j} \geqslant 1$ and it is equal to $p_j^{k_j}$ with $k_j < k$, if $\mu_{1,j} = 0$. Hence k is the greatest of the numbers $k_1, k_2, \ldots, k_r$ such that

$$\omega\big((Z)\big):\omega\big(p_j\cdot(Z)\big) = p_j^{k_j}.$$

Consequently, k is uniquely determined by the power domain (Z).

If $k = 1$, then the system $(m_1, m_2, \ldots, m_k)$ consists of only one number m_1 and is determined by (Z). Assume that $k = n+1 > 1$ and that for $k \leqslant n$ the system $(m_1, m_2, \ldots, m_k)$ is determined by (Z).

Let l_j denote, for $j = 1, 2, \ldots, r$ the number of exponents $\mu_{i,j}$ equal to $\mu_{1,j}$. Hence $1 \leqslant l_j \leqslant k$. It follows by (6.1) that if $\mu = \mu_{l,j}$ with $l \leqslant l_j$ (that is if $\mu = \mu_{1,j}$), then the quotient $\omega\big((Z)\big):\omega\big(p_j^{\mu}\cdot(Z)\big)$ is equal to $p_j^{k\cdot\mu}$. If, however, $\mu > \mu_{1,j}$, then this quotient is less than $p_j^{k\cdot\mu}$. Thus l_j and $\mu_{1,j}$ (for $j = 1, 2, \ldots, r$) depend only on the power domain (Z). Hence the number $m_1 = p_1^{\mu_{1,1}}\cdot p_2^{\mu_{1,2}}\ldots p_r^{\mu_{1,r}}$ is determined by (Z) and the same is with the numbers $s = \min\limits_{j=1,2,\ldots,r} l_j$ of elements equal to m_1 in the system $(m_1, m_2, \ldots, m_k)$.

Now let us observe that $m_1\cdot(Z)$ is isomorphic to the domain of the group $\mathfrak{N}_{m_{s+1}/m_1}\times\ldots\times\mathfrak{N}_{m_k/m_1}$. By the induction hypothesis, the system of numbers $(m_{s+1}/m_1, \ldots, m_k/m_1)$ is determined by the power domain $m_1\cdot(Z)$, hence also by the power domain (Z). It follows that also the system $(m_1, m_2, \ldots, m_k)$, where $m_1 = m_2 = \ldots = m_s$, is determined by the power domain (Z). Thus the proof of theorem (6.2) is finished.

For every group Z, let us denote by $\mathscr{T}(Z)$ the *torsion* of Z, that is the subgroup of Z consisting of all elements of finite order. We get from theorem (6.2) the following

(6.7) COROLLARY. *If Z, Z' are two abelian groups with isomorphic power domains and if $\mathscr{T}(Z)$ is finite, then the groups $\mathscr{T}(Z)$ and $\mathscr{T}(Z')$ are isomorphic.*

Proof. It is clear that $(\mathscr{T}(Z))$ coincides with the subdomain $(Z)_0$ of the power domain (Z) consisting of all elements of finite order, and that $(\mathscr{T}(Z'))$ coincides with the subdomain $(Z')_0$ of (Z') consisting of all elements of finite order. Since (Z) and (Z') are isomorphic, we infer that $(Z)_0$ and $(Z')_0$ are isomorphic. It follows, by theorem (6.2), that $\mathscr{T}(Z)$ is isomorphic with $\mathscr{T}(Z')$.

7. Power domains of finitely generated abelian groups. Let us prove the following

(7.1) THEOREM. *The power domains $(Z), (Z')$ of two finitely generated abelian groups Z, Z' are isomorphic if and only if one of the two following conditions are satisfied:*

(*) *Z and Z' are isomorphic.*

(**) *The torsion groups $\mathscr{T}(Z), \mathscr{T}(Z')$ are isomorphic and the ranks of Z and of Z' are greater than* 1.

Proof. It is clear that (*) implies $(Z) \simeq (Z')$. If the condition (**) is satisfied, then (see [3], p. 308) there exist two integers $k, l > 1$ such that

$$Z \simeq \mathfrak{N}^k \times \mathscr{T}(Z) \quad \text{and} \quad Z' \simeq \mathfrak{N}^l \times \mathscr{T}(Z), \tag{7.2}$$

and we infer by (4.3) that $(Z) \simeq (Z')$.

On the other hand, if $(Z) \simeq (Z')$, then we infer by corollary (6.7) that $\mathscr{T}(Z) \simeq \mathscr{T}(Z')$, hence Z and Z' satisfy (7.2). If $k = l$, then the condition (*) is satisfied. If $k \neq l$, then we infer by (4.3) and (7.2) that $k, l > 1$, that is the condition (**) is satisfied. Thus the proof of theorem (7.1) is finished.

REFERENCES

[1] K. Borsuk, *Theory of retracts*, Monografie Matematyczne 44, Warszawa 1967.
[2] G. Grätzer, *Universal algebra*, Princeton 1968.
[3] H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Leipzig — Berlin 1934.

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