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ON THE TOTAL POSITIVITY OF THE TRUNCATED POWER KERNEL

BY

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A kernel $K(x,t)$ is said to be *totally positive* on $X \times T$ if there is an $\varepsilon = +1$ or $\varepsilon = -1$ such that

$$(1) \qquad \varepsilon \det\{K(x_i,t_j)\}_{i=1,\, j=1}^{N,\ N} \geq 0$$

for each choice of the points $x_1 < \ldots < x_N$ in X and $t_1 < \ldots < t_N$ in T. The truncated power kernel

$$(x-t)_+^{r-1} := \begin{cases} (x-t)^{r-1} & \text{if } x-t \geq 0, \\ 0 & \text{if } x-t < 0, \end{cases}$$

is totally positive on $X \times T$ for each $X, T \subset \mathbf{R}$ (see Karlin [4]). This fact plays a fundamental role in the theory of spline functions. The purpose of this note is to show that the relation (1) remains true in a more general setting involving Birkhoff type matrices $\{K(x_i,t_j)\}$.

1. Preliminaries. B-splines with Birkhoff knots. Consider a pair $(\mathbf{x}, E)$ with $\mathbf{x} = (x_i)_{i=1}^m$, $x_1 < \ldots < x_m$, and with an incidence matrix $E = (e_{ij})_{i=1,\, j=0}^{m,\ r-1}$. Denote by $|E|$ the number of 1-entries in E.

We shall say that the pair $(\mathbf{x}, E)$ is *regular* (respectively, *s-regular*) if:

(i) E is conservative;

(ii) E satisfies the Pólya condition (respectively, the strong Pólya condition).

All notions used above are well known in the theory of Birkhoff interpolation (see [5] for details).

Let $(\mathbf{x}, E)$ be a regular pair with $|E| = r+1$. Then, by the Atkinson–Sharma theorem [1], the Birkhoff interpolation problem

$$(2) \qquad p^{(j)}(x_i) = f^{(j)}(x_i) \quad \text{if} \quad e_{ij} = 1$$

has a unique solution $p_f \in \pi_r$ (π_n denotes the set of all algebraic polynomials of degree n). Equivalently, there exists a unique linear functional

$$D[(\mathbf{x}, E); f] = \sum_{e_{ij}=1} c_{ij} f^{(j)}(x_i)$$

satisfying the conditions:

$$D[(\mathbf{x},E);f]=0 \quad \text{for} \quad f(x)=x^k\,, \quad k=0,\dots,r-1\,,$$
$$D[(\mathbf{x},E);f]=1 \quad \text{for} \quad f(x)=x^r\,.$$

$D[(\mathbf{x},E);f]$ is called the *divided difference* of f at $(\mathbf{x},E)$. Note that $D[(\mathbf{x},E);f]$ coincides with the coefficient of x^r in the polynomial p_f which interpolates f at $(\mathbf{x},E)$, i.e., which satisfies (2).

LEMMA 1. *Suppose that the pair $(\mathbf{x},E)$ is regular, $\mathbf{x}=(x_1,\dots,x_m)$, $E=(e_{ij})_{i=1,\,j=0}^{m,\ r}$ and $|E|=r+1$. Then $c_{m\lambda}>0$, where λ is the order of the highest derivative of f at x_m, appearing in the expression $D[(\mathbf{x},E);f]$.*

Proof. Let φ be the polynomial from π_r that satisfies the interpolation conditions

$$\varphi^{(j)}(x_i)=\delta_{im}\delta_{j\lambda} \quad \text{if} \quad e_{ij}=1\,.$$

Then

$$c_{m\lambda}=D[(\mathbf{x},E);\varphi]\,.$$

On the other hand, $D[(\mathbf{x},E);\varphi]$ is the coefficient C of x^r in the polynomial φ. Thus

$$\operatorname{sign} c_{m\lambda}=\operatorname{sign} C=\operatorname{sign}\varphi^{(r)}(x_m)\,. \tag{3}$$

Now a very careful study of the behaviour of the sign changes in the sequence $\varphi(x),\varphi'(x),\dots,\varphi^{(r)}(x)$ when x runs from $a:=x_1$ to $b:=x_m$ shows that $\varphi^{(\lambda)}(b),\dots,\varphi^{(r)}(b)$ does not contain a sign change. Therefore $\operatorname{sign}\varphi^{(r)}(b)=\operatorname{sign}\varphi^{(\lambda)}(b)=1$, which, in view of (3), completes the proof.

For regular $(\mathbf{x},E)$ with $|E|=r+1$, the function

$$B[(\mathbf{x},E);t]:=D[(\mathbf{x},E);(\,\cdot\,,-t)_+^{r-1}]$$

is said to be a *B-spline* of degree $r-1$ with knots $(\mathbf{x},E)$. This natural extension of the original Curry–Schoenberg B-splines was introduced and studied in [2]. Many of the crucial properties of the extended B-splines $B[(\mathbf{x},E);t]$ were proved there. It is known, for instance, that $B[(\mathbf{x},E);t]$ has a finite support and does not change sign on $\mathbf{R}$. This could be derived from a general theorem about the number of zeros of polynomial splines with Birkhoff knots (see Theorem 7.13 in [5]). We give here a new, simple direct proof of this fact.

PROPOSITION 1. *Let a pair $(\mathbf{x},E)$ be s-regular and $|E|=r+1$. Then*

$$B[(\mathbf{x},E);t]=0 \quad \textit{for} \quad t\notin[x_1,x_m]\,, \tag{5}$$
$$B[(\mathbf{x},E);t]>0 \quad \textit{for} \quad t\in(x_1,x_m)\,. \tag{6}$$

Proof. The equality (5) is clear since the function $g(x) := (x - t)_+^{r-1}$ vanishes on (x_1, x_m) for $t > x_m$ and g coincides on (x_1, x_m) with the polynomial $(x - t)^{r-1}$ for each fixed $t < x_1$.

Let us prove (6). According to the remark after the definition of the divided difference, $B(t) := B[(\mathbf{x}, E); t]$ is the coefficient of x^r in the polynomial $p \in \pi_r$ which interpolates g at $(\mathbf{x}, E)$. Note that $p \not\equiv 0$ and $p \not\equiv g$ in (x_1, x_m), and consequently, in any subinterval of (x_1, x_m). Therefore $p(x) - g(x)$ has only isolated zeros in (x_1, x_m). Since $p(x) - g(x)$ vanishes at $(\mathbf{x}, E)$, we see by Rolle's theorem and the s-regularity assumption that $p^{(r-1)}(x) - g^{(r-1)}(x)$ must have at least two sign changes in (x_1, x_m). This is possible only if $p^{(r-1)}(x)$ is an increasing linear function, i.e., if p has positive leading coefficient $B(t)$. This completes the proof.

Our further considerations are based on the total positivity of a certain matrix of the form $\{B[(\mathbf{x}_i, E_i); t_j]\}$. In order to formulate the result we need some definitions.

Given an integer $r > 0$ and a pair $(\mathbf{x}, E)$ such that $x_1 < \ldots < x_m$, $E = (e_{ij})_{i=1,\, j=0}^{m,\; r-1}$, $|E| = r + N$, we defined in [2] the $(r + 1)$-partition of $(\mathbf{x}, E)$ to be a sequence of pairs $\{(\mathbf{x}_i, E_i)\}_{i=1}^{N}$ obtained from $(\mathbf{x}, E)$ in the following way. Order the elements of E row by row, i.e., in the manner $e_{10}, \ldots, e_{1,r-1}, \ldots, e_{m0}, \ldots, e_{m,r-1}$ and number the 1-entries in this sequence from 1 to $r+N$. Let $\mathbf{e}_p, \mathbf{e}_{p+1}, \ldots, \mathbf{e}_q$ be the rows of E which contain $r + 1$ consecutive 1-entries starting from the ith one. Suppose that the first row $\mathbf{e}_p$ (respectively, the last row $\mathbf{e}_q$) contains n_1 (respectively, n_2) 1-entries of this $(r + 1)$-sample. We denote by E_i the matrix $\{\mathbf{e}_p, \ldots, \mathbf{e}_q\}$ in which all 1's in the sequence $e_{p0}, \ldots, e_{p,r-1}$ (respectively, in $e_{q0}, \ldots, e_{q,r-1}$) except the first n_1 (respectively, n_2) are replaced by 0. Finally, define $\mathbf{x}_i := (x_p, \ldots, x_q)$.

We say that the $(r+1)$-partition $\{(\mathbf{x}_i, E_i)\}$ of $(\mathbf{x}, E)$ is s-regular if each $(\mathbf{x}_i, E_i)$ is s-regular.

PROPOSITION 2. *Let* $\mathbf{x} = (x_0, \ldots, x_{m+1})$, $x_0 < x_1 < \ldots < x_{m-1}$, $E = (e_{ij})_{i=0,\, j=0}^{m+1,\; r-1}$ *and* $|E| = r + N$. *Suppose that the* $(r + 1)$*-partition* $\{(\mathbf{x}_k, E_k)\}_{k=1}^{N}$ *of* $(\mathbf{x}, E)$ *is s-regular. Then*

$$\Delta := \det\{B_k(\tau_j)\}_{k=1,\, j=1}^{N,\; N} \geq 0 \tag{7}$$

for any $\tau_1 \leq \ldots \leq \tau_N$ *satisfying* $\tau_j < \tau_{j+r}$, $j = 1, \ldots, N - r$, *where* $B_k := B[(\mathbf{x}_k, E_k);\ \cdot\ ]$. *Moreover,* Δ *is positive if and only if*

$$\tau_k \in \operatorname{supp} B_k\,, \quad k = 1, \ldots, N\,.$$

This extension of the Schoenberg–Whitney theorem (see [6]) was proved in [2].

2. Main result. A spline function of degree $r-1$ with knots $\xi_1 < \ldots < \xi_n$ of respective multiplicities $\nu_1, \ldots, \nu_n$ is any expression of the form

$$s(x) = p(x) + \sum_{k=1}^{n} \sum_{\lambda=0}^{\nu_k - 1} a_{k\lambda}(x - \xi_k)_+^{r-\lambda-1}$$

where $\{a_{k\lambda}\}$ are real constants and $p \in \pi_{r-1}$.

Let $(\mathbf{x}, E)$ be a given pair with $\mathbf{x} = (x_0, \ldots, x_{m+1})$, $a = x_0 < x_1 < \ldots < x_{m+1} = b$, and with an incidence matrix $E = (e_{ij})_{i=0,}^{m+1}\,_{j=0}^{r-1}$ such that $|E| = r + N$. Consider the Birkhoff interpolation problem

$$s^{(j)}(x_i) = f_{ij} \quad \text{if} \quad e_{ij} = 1, \tag{8}$$

where $\{f_{ij}\}$ are given values and $p(x)$ is written in the form

$$p(x) = a_0 + a_1(x-a) + \ldots + a_{r-1}(x-a)^{r-1}.$$

In what follows we define $s^{(j)}(x)$ as $s^{(j)}(x+0)$ in case $s^{(j)}$ is discontinuous at x. Denote by $V = V[(\mathbf{x}, E), (\xi, \nu)]$ the matrix of the system (8) with respect to the unknowns

$$a_0, \ldots, a_{r-1}, a_{10}, \ldots, a_{1,\nu_1-1}, \ldots, a_{n0}, \ldots, a_{n,\nu_n-1}.$$

We shall show that

$$\varepsilon \det V[(\mathbf{x}, E), (\xi, \nu)] \geq 0$$

for each $\mathbf{x}$ and ξ with some $\varepsilon = (-1)^\sigma$ where σ depends only on the structure of E. In a fairly general situation, including quasi-Hermitian E, we find the explicit value of σ and thus provide a new proof of a fundamental result of S. Karlin [4].

We start with an auxiliary lemma.

Denote, for simplicity, by

$$\begin{bmatrix} \{u_1(t), \ldots, u_n(t)\}^{(j)}|_{t=t_i} \\ e_{ij} = 1, \quad e_{ij} \in E \end{bmatrix}$$

the matrix consisting of the rows

$$u_1^{(j)}(t_i), \ldots, u_n^{(j)}(t_i)$$

ordered according to the position of the 1-entries e_{ij} in the sequence of consecutive rows of the incidence matrix $E = (e_{ij})$.

LEMMA 2. *Let $(\mathbf{y}, G)$ be a given pair with $\mathbf{y} = (y_1, \ldots, y_k)$ and with an incidence matrix $G = (g_{ij})_{i=1,}^{k}\,_{j=0}^{r-1}$ such that $|G| = r$. Let*

$$A = \begin{bmatrix} \{1, x-a, \ldots, (x-a)^{r-1}\}^{(j)}|_{x=y_i} \\ g_{ij} = 1, \quad g_{ij} \in G \end{bmatrix}.$$

Suppose that $(\mathbf{y}, E)$ *is a regular pair. Then there is a positive integer* σ *depending only on* G *such that*

$$(-1)^\sigma \det A > 0$$

for each $a \le y_1 < \ldots < y_k$. *Moreover, if the* 1*-entries of* $\mathbf{g}_i := (g_{i0}, \ldots, g_{i,r-1})$ *remain in the lowest* $|\mathbf{g}_i|$ *positions of the* 1*-entries in the coalescence of* $\mathbf{g}_i$ *and* $\mathbf{g}_{i+1}$, *for* $i = 1, \ldots, k-1$, *then* $\sigma = 0$, *and if this holds for* $i = 2, \ldots, k-1$, *then*

$$\sigma = p(p+1)/2 + i_1 + \ldots + i_p,$$

where $i_1, \ldots, i_p$ *are the positions of* 1*'s in* $\mathbf{g}_1$.

Proof. Since $(\mathbf{y}, G)$ is a regular pair, by the Atkinson–Sharma theorem [1], the interpolation problem

$$\{a_0 + a_1(x-a) + \ldots + a_{r-1}(x-a)^{r-1}\}^{(j)}|_{x=y_i} = 0 \quad \text{if} \quad g_{ij} = 1$$

has a unique solution. Thus $\det A \neq 0$ for each $a \le y_1 < \ldots < y_k$. One can even find the sign of $\det A$. In order to do this, note that for fixed $y_1, \ldots, y_{k-1}$, $\det A$ is a polynomial function of $x := y_k - y_{k-1}$. Denote this function by $A_k(x)$. By Taylor's formula,

$$A_k(x) = \sum_{j=0} A_k^{(j)}(0)x^j/j!. \tag{9}$$

Let $A_k^{(\lambda)}(0)$ be the first nonzero coefficient in (9). It is not difficult to see that $A_k^{(\lambda)}(0)$ is equal (up to a positive integer factor) to a determinant A_{k-1} that is obtained from $A_k(x)$ by replacing its last $n := |\mathbf{g}_k|$ rows with rows of the form

$$\{1, x-a, \ldots, (x-a)^{r-1}\}^{(j)}|_{x=y_{k-1}}$$

for $j = j_1, \ldots, j_n$, where $j_1, \ldots, j_n$ are the positions of the first n 0-entries in the sequence $(g_{k-1,\mu}, \ldots, g_{k-1,r-1})$, μ being the position of the first 1-entry in $\mathbf{g}_k$. Clearly,

$$\operatorname{sign} A_k(x) = \operatorname{sign} A_{k-1}$$

for sufficiently small $x > 0$.

Now A_{k-1} is a determinant corresponding to $y_1 < \ldots < y_{k-1}$ and an incidence matrix G_{k-1} which is obtained from G by coalescence of the last two rows $\mathbf{g}_{k-1}$ and $\mathbf{g}_k$.

Repeating this procedure with respect to A_{k-1} we get A_{k-2}, and so on. Finally, we come to the relation

$$\operatorname{sign} A_k(x) = \operatorname{sign} A_1,$$

where A_1 is a Taylor matrix

$$\begin{bmatrix} \{1, x-a, \ldots, (x-a)^{r-1}\}^{(j)}|_{x=a} \\ j = j_0, \ldots, j_{r-1} \end{bmatrix}$$

with $(j_0,\dots,j_{r-1})$ a certain permutation of $(0,\dots,r-1)$. Thus

$$\operatorname{sign}\det A = (-1)^\sigma\,,$$

where σ is the number of transpositions needed to rearrange the numbers $(j_0,\dots,j_{r-1})$ in the natural order.

It is easily seen that $\sigma = 0$, i.e., $(j_0,\dots,j_{r-1}) = (0,\dots,r-1)$, if the assumption of the lemma holds for $i = 1,\dots,k-1$. For example, this clearly holds if $\mathbf{g}_i$ contains only one block $\beta_i := [g_{i,l},\dots,g_{i,l+q}]$ of 1-entries (l is the *level* of β_i) for $i = 1,\dots,k$ and the level increases or remains the same when i increases. This condition holds for Hermitian matrices G.

Another particular case: if the previous assumption holds for $i = 2,\dots,k$ and $i_1,\dots,i_p$ are the positions of the 1-entries in $\mathbf{g}_1$, then

$$(j_0,\dots,j_{r-1}) \equiv (i_1,\dots,i_p,k_1,\dots,k_{r-p})\,,$$

where $k_1 < \ldots < k_{r-p}$ and thus

$$\sigma = (i_1-1)+(i_2-2)+\ldots+(i_p-p) = p(p+1)/2 + i_1 + \ldots + i_p\,.$$

The lemma is proved.

THEOREM 1. *Let* $\mathbf{x} = (x_0,x_1,\dots,x_{m+1})$, $a = x_0 < x_1 < \ldots < x_{m+1} = b$, $E = (e_{ij})_{i=0,\ j=0}^{m+1\ r-1}$ *and* $|E| = r+N$. *Suppose that* $\{(\mathbf{x}_k,E_k)\}_{k=1}^N$ *is an s-regular* $(r+1)$*-partition of* $(\mathbf{x},E)$. *Then there is a* σ, *depending only on* E, *such that*

$$(-1)^\sigma \det V[(\mathbf{x},E),(\xi,\nu)] \ge 0$$

for each choice of the set

$$\xi = (\tau_1,\dots,\tau_N) \equiv ((\xi_1,\nu_1),\dots,(\xi_n,\nu_n))$$

of points $\xi_1 < \ldots < \xi_n$ *with respective multiplicities* $\nu_1,\dots,\nu_n$ *such that* $1 \le \nu_i \le r$, $i = 1,\dots,n$, $\nu_1+\ldots+\nu_n = N$. *Moreover,*

$$(-1)^\sigma \det V[(\mathbf{x},E),(\xi,\nu)] > 0$$

if and only if $\tau_i \in \operatorname{supp} B[(\mathbf{x}_i,E_i);t]$, $i = 1,\dots,N$.

Proof. Clearly the matrix $V[(\mathbf{x},E),(\xi,\nu)]$ consists of the rows

$$\mathbf{w}_{ij} := \{1,(x-a),\dots,(x-a)^{r-1},K(x,\xi_1),\dots,K^{(\nu_n-1)}(x,\xi_n)\}^{(j)}|_{x=x_i}$$

where (i,j) runs over the indices of all 1-entries e_{ij} in the sequence

$$e_{00},\dots,e_{0,r-1},\dots,e_{10},\dots,e_{1,r-1},\dots,e_{m+1,0},\dots,e_{m+1,r-1}$$

and $K(x,t) := (x-t)_+^{r-1}$, $K^{(j)}(x,t) := (\partial^j/\partial t^j)K(x,t)$. In order to find $\det V$ we shall perform some elementary transformations in V, writing in row $r+k$ $(k = 1,\dots,N)$ a linear combination of rows

$$\mathbf{v}_{r+k} := \sum_{e_{ij}=1} c_{ij}\mathbf{w}_{ij}$$

where the sum is over the 1-entries of E_k and $\{c_{ij}\}$ are the coefficients in the divided difference

$$D[(\mathbf{x}_k, E_k); f] = \sum_{e_{ij}=1} c_{ij} f^{(j)}(x_i) .$$

Denote by α_k the coefficient of the highest derivative at the last point of $\mathbf{x}_k$ appearing in $D[(\mathbf{x}_k, E_k); f]$. According to Lemma 1,

$$\alpha_k > 0 . \tag{10}$$

Denote by V_0 the matrix obtained from V by the described transformation of rows $r+1, \ldots, r+N$. Clearly $\det V = \alpha \det V_0$ with $\alpha := 1/(\alpha_1 \ldots \alpha_N)$, and the $(r+k)$th row of V is of the form

$$\mathbf{v}^0_{r+k} := \{D_k[1], \ldots, D_k[(x-a)^{r-1}], D_k[K(x,\xi_1)], \ldots, D_k[K^{(\nu_n - 1)}(x, \xi_n)]\}$$

where $D_k := D[(\mathbf{x}_k, E_k);\ \cdot\ ]$. Using the property $D_k[f] = 0$ for all $f \in \pi_{r-1}$ and the definition of B-splines we see that

$$\mathbf{v}^0_{r+k} = \{0, \ldots, 0, B_k(\xi_1), \ldots, B_k^{(\nu_n - 1)}(\xi_n)\} .$$

Let $i_1, \ldots, i_p$ be the positions of the 1-entries in $(e_{00}, \ldots, e_{0,r-1})$. Then, by the Laplace formula,

$$\det V = \alpha \det A \cdot \det\{B_k(\tau_j)\}_{k=1,\, j=1}^{N\quad N} \tag{11}$$

where

$$A = \begin{bmatrix} \{1, x-a, \ldots, (x-a)^{r-1}\}^{(j)}|_{x=x_i} \\ e_{ij} = 1, \quad e_{ij} \in E_0 \end{bmatrix}$$

and E_0 is obtained from E_1 by replacing the last 1-entry (i.e., the last 1 in the last row of E_1) by 0. Since E_1 was assumed to be s-regular, E_0 is regular. Then, by Lemma 2, $\det A \neq 0$. Further, by Proposition 2,

$$\Delta := \det\{B_k(\tau_j)\}_{k=1,\, j=1}^{N\quad N} \geq 0$$

and strict inequality holds if and only if $\tau_k \in \operatorname{supp} B_k$, $k = 1, \ldots, N$. Therefore, in view of (10) and (11), $\det V \neq 0$ if and only if $\Delta \neq 0$, and

$$\operatorname{sign} \det V = \operatorname{sign} \det A . \tag{12}$$

The theorem is proved.

Next we derive Karlin's total positivity theorem as a particular case of Theorem 1.

COROLLARY 1. *Let* $(\mathbf{x}, E)$ *be any pair with* $\mathbf{x} = (x_0, x_1, \ldots, x_{m+1})$, $a = x_0 < x_1 < \ldots < x_{m+1} = b$, *and with a quasi-Hermitian incidence matrix* $E = (e_{ij})_{i=0,\ j=0}^{m+1\ \ r-1}$ *such that* $|E| = r + N$. *Suppose that* $\{(\mathbf{x}_k, E_k)\}_{k=1}^N$ *is an* *s-regular $(r+1)$-partition of $(\mathbf{x}, E)$. Then*

$$\det\begin{bmatrix}\{1,(x-a),\dots,(x-a)^{r-1},K(x,\xi_1),\dots,K^{(\nu_n-1)}(x,\xi_n)\}^{(j)}|_{x=x_i}\\ e_{ij}=1,\quad e_{ij}\in\hat{E}\end{bmatrix}\geq 0$$

for each choice of points $\xi_1 < \dots < \xi_n$ with respective multiplicities $\nu_1,\dots,\nu_n$ such that $1 \leq \nu_i \leq r$, $i=1,\dots,n$, $\nu_1+\dots+\nu_n = N$. Here $\hat{E}$ is the matrix obtained from E by replacing the first r 1-entries by 0 (i.e., annihilating the matrix E_0). Strict inequality holds if and only if

$$\tau_i \in \operatorname{supp} B[(\mathbf{x}_i, E_i); t], \quad i=1,\dots,N.$$

Proof. Denote the determinant considered by W. Clearly, up to a positive constant,

$$W = (-1)^\sigma \det V = (-1)^\sigma \det A \cdot \det\{B_k(\tau_j)\},$$

where σ and A are as in the theorem. Since E_1 is quasi-Hermitian, sign det A $= (-1)^\sigma$ and the assertion follows from Theorem 1.

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