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ABSTRACT EXTERIOR DIFFERENTIAL

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As it is known (see [1] and [2]), a big part of the classical differential geometry belongs to linear algebra, more precisely, to the theory of modules over a commutative ring $\mathscr{R}$. A purely algebraic theory can be developped in a way such that, in the case where $\mathscr{R}$ is the ring of all real smooth (i.e., of class C^∞) functions on a smooth manifold M, we get many fundamental theorems from the differential geometry on M. In particular, the theory of exterior forms (i.e., of multi-linear skew-symmetric mappings) can be treated in such an algebraic way (for details, see [1]).

The aim of the present paper is to give a purely algebraic proof of the formula for the second exterior differential, i.e., the formula establishing a connexion between the second exterior differential and the curvature tensor. The proof given in this paper differs from the algebraic proof given in [1]. The degree of generality is here bigger than in [1] or [2]. The notion of a module is replaced by that of an abelian group.

1. Skew-symmetric mappings. In this paper *group* means always *abelian group* written additively and *additive mapping* means *homomorphism*. If V is a set, then V^n denotes the Cartesian product $V \times V \times \ldots \times V$ (n times).

Suppose that W_1, W_2 and W are groups and that there is defined a multiplication

(1) $$w_1 w_2 \in W \quad \text{for } w_1 \in W_1 \text{ and } w_2 \in W_2,$$

that is, a bi-additive mapping form $W_1 \times W_2$ into W. The value at a pont $(w_1, w_2) \in W_1 \times W_2$ will be denoted by $w_1 w_2$ (by hypothesis, $(w_1 \pm w_1')w_2 = w_1 w_2 \pm w_1' w_2$, and the dual identity is also true). Then, for any set V and for any skew-symmetric mappings $L_1 : V^r \to W_1$ and $L_2 : V^s \to W_2$, we can define the *exterior product* $L_1 \wedge L_2 : V^{r+s} \to W$ which also is skew-symmetric. Namely,

$$L_1 \wedge L_2(v_1, \ldots, v_{r+s})$$
$$= \sum \operatorname{sgn}(i_1, \ldots, i_{r+s}) L_1(v_{i_1}, \ldots, v_{i_r}) L_2(v_{i_{r+1}}, \ldots, v_{i_{r+s}}) \quad \text{for } v_1, \ldots, v_{r+s} \in V,$$

where $\sum$ is extended over all permutations $i_1, \ldots, i_{r+s}$ of integers $1, \ldots$ $\ldots, r+s$ such that $i_1 < i_2 < \ldots < i_r$, $i_{r+1} < i_{r+2} < \ldots < i_{r+s}$, and $\mathrm{sgn}(\ldots)$ is the sign of permutation.

In this paper we consider only multiplication (1) of one of the following two types:

1° W_1 is a group of homomorphisms $w_1: W_2 \to W$ and the multiplication is defined by

$$w_1 w_2 = w_1(w_2) \quad \text{for } w_1 \in W_1 \text{ and } w_2 \in W_2.$$

2° W_1, W_2 and W are groups of all endomorphisms in a given group W_0. Multiplication (1) is the composition

$$w_1 w_2 = w_1 \circ w_2 \quad \text{for } w_1 \in W_1 \text{ and } w_2 \in W_2.$$

1.1. *For every mapping J from a set V into the group of all endomorphisms of a group W and for every skew-symmetric mapping $L: V^n \to W$ the following identity holds:*

$$J \wedge (J \wedge L) = (J \wedge J) \wedge L.$$

Here

$$J \wedge J(v_1, v_2) = J(v_1)\circ J(v_2) - J(v_2)\circ J(v_1) \quad \text{for } v_1, v_2 \in V, \tag{2}$$

according to 2°.

Suppose now that V and W are groups. A mapping $L: V^n \to W$ is said to be *multi-additive* or, more precisely, *n-additive* if the expression $L(v_1, \ldots, v_n)$ is an additive function of each variable $v_1, \ldots, v_n$ separately.

If $L: V^n \to W$ is skew-symmetric and n-additive and $S: V^m \to V$ is skew-symmetric and m-additive, then the formula

$$S * L(v_1, \ldots, v_{n+m-1}) \\ = \sum \mathrm{sgn}(i_1, \ldots, i_{n+m-1}) L\big(S(v_{i_1}, \ldots, v_{i_m}), v_{i_{m+1}}, \ldots, v_{i_{n+m-1}}\big),$$

where $\sum$ is extended over all permutations $i_1, \ldots, i_{n+m-1}$ of integers $1, \ldots, n+m-1$ such that $i_1 < i_2 < \ldots < i_m$ and $i_{m+1} < i_{m+2} < \ldots < i_{m+n-1}$, and sgn $(\ldots)$ is the sign of permutation, defines a skew-symmetric and $(n+m-1)$-additive mapping $S*L: V^{n+m-1} \to W$. The most important case is that of $m = 2$. If $S: V^2 \to V$ is 2-additive and skew-symmetric and $L: V^n \to W$ is n-additive and skew-symmetric, then the skew-symmetric $(n+1)$-additive mapping $S*L: V^{n+1} \to W$ is defined by

$$S * L(v_1, \ldots, v_{n+1}) \\ = -\sum_{i<j} (-1)^{i+j} L\big(S(v_i, v_j), v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{n+1}\big).$$

In particular,

(3) $S*S(v_1, v_2, v_3) = S(S(v_1, v_2), v_3)+S(S(v_2, v_3), v_1)+S(S(v_3, v_1), v_2)$.

1.2. *If $S\colon V^2 \to V$ is skew-symmetric and 2-additive, and $L\colon V^n \to W$ is skew-symmetric and n-additive, then $S*(S*L) = (S*S)*L$.*

1.3. *If $S\colon V^2 \to V$ is skew-symmetric and 2-additive, and $L\colon V^n \to W$ is skew-symmetric and n-additive, then for every homomorphism J from ihe group V into the group of all endomorphisms of the group W the followtng identity holds:*

$$S*(J \wedge L)+J \wedge (S*L) = (S*J) \wedge L.$$

Simple proofs of lemmas 1.1-1.3 are omitted. The lemmas are particular cases of certain more general statements which are not quoted here.

2. Exterior differential and curvature tensor. In this section V is a fixed group and S is a *Lie multiplication* in V, i.e., $S\colon V^2 \to V$ is 2-additive and skew-symmetric and, moreover, the following *Jacobi identity* holds (see (3)):

(4) $$S*S = 0.$$

In this section W is another fixed group and J is a fixed homomorphism from the group V into the group of all endomorphisms of the group W. The mapping J will be called a *covariant derivative in* W. By definition, for every $v \in V$, $J(v)\colon W \to W$ is an additive mapping.

By the *curvature tensor* of a covariant derivative J we mean a mapping R which assigns, to any $v_1, v_2 \in V$, the additive mapping (homomorphism)

$$R(v_1, v_2) = J(v_1)\circ J(v_2)-J(v_2)\circ J(v_1)-J(S(v_1, v_2))\colon W \to W.$$

By the definition (see (2)), $R = J \wedge J-S*J$.

For any n-linear skew-symmetric mapping $L\colon V^n \to W$, by the *exterior differential* of L we mean the $(n+1)$-additive skew-symmetric mapping $dL\colon V^{n+1} \to W$ defined by $dL = J \wedge L-S*L$.

By the definition,

$$dL(v_1, \ldots, v_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1}J(v_i)\big(L(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n+1})\big)+$$
$$+\sum_{i<j} L\big(S(v_i, v_j), v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{n+1}\big).$$

2.1. *For every n-additive skew-symmetric mapping $L\colon V^n \to W$ the identity $ddL = R \wedge L$ holds, that is,*

$$ddL(v_1, \ldots, v_{n+2})$$
$$= -\sum_{i<j} (-1)^{i+j}R(v_i, v_j)\big(L(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{n+2})\big)$$

for any $v_1, \ldots, v_{n+2} \in V$.

Indeed, by lemmas 1.1-1.3 and identity (4),

$$\begin{aligned} ddL &= J \wedge dL - S * dL = J \wedge (J \wedge L - S * L) - S * (J \wedge L - S * L) \\ &= J \wedge (J \wedge L) - J \wedge (S * L) - S * (J \wedge L) + S * (S * L) \\ &= (J \wedge J) \wedge L - (S * J) \wedge L + (S * S) * L = (J \wedge J - S * J) \wedge L = R \wedge L. \end{aligned}$$

To apply theorem 2.1 to differential geometry, let us assume that M is a smooth manifold and $\mathscr{R}$ is the ring of all smooth functions on M. Let V be the module (over $\mathscr{R}$) of all smooth tangent vector fields on M, and let S be the ordinary Lie product, $S(v_1, v_2) = [v_1, v_2]$. Let W be any module (over $\mathscr{R}$) appearing in the differential geometry on M, for instance, the module V of all smooth tangent vector fields on M, or the module of all smooth tangent covector fields on M, or the module of all smooth tangent tensor fields (of a fixed type) on M, etc. Let J be an ordinary covariant derivative in the module W, i.e., a module-linear mapping from V into the set of all linear mappings from W into W (expression $J(v)(w)$, denoted as a rule by the symbol $\nabla_v w$ $(v \in V, w \in W)$, satisfies well-known conditions). Then theorem 1.2 yields the well-known formula for the second exterior derivative. We can also assume that W is the ring $\mathscr{R}$ and that $J(v)(a)$ is the directional derivative of the function $a \in \mathscr{R}$ in the direction $v \in V$. The curvature tensor of this covariant derivative in $\mathscr{R}$ is equal to 0. In this case theorem 2.1 yields the known formula $ddL = 0$ for differential forms L on M.

REFERENCES

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