COLLOQUIUM MATHEMATICUM

XXVI DÉDIÉ À M. WŁADYSŁAW ŚLEBODZIŃSKI 1972

RIEMANNIAN MANIFOLDS ADMITTING CERTAIN CONFORMAL CHANGES OF METRIC

BY

CHUAN-CHIH HSIUNG (BETHLEHEM, PA.) AND LARRY R. MUGRIDGE (EASTON, PA.)

1. Introduction. Let M be an n-dimensional differentiable connected Riemannian manifold with metric tensor g. In order to distinguish g from other metrics on M, we denote the manifold M with g by (M, g). If g^* is another metric on M, and there is a function σ on M such that $g^* = e^{2\sigma}g$, then g and g^* are said to be *conformally related* or *conformal to each other*, and such a change of metric $g \to g^*$ is called a *conformal change* of Riemannian metric.

Let (M, g) and (M', g') be two Riemannian manifolds, and $f\colon M \to M'$ a diffeomorphism. Then $g^* = f^{-1}g'$ is a Riemannian metric on M. When g^* and g are conformally related, that is, when there exists a function σ on M such that $g^* = e^{2\sigma}g$, we call $f\colon (M, g) \to (M', g')$ a *conformal transformation*. In particular, if σ is constant, then f is called a *homothetic transformation* or a *homothety*; if $\sigma = 0$, then f is called an *isometric transformation* or an *isometry*. The group of all conformal (homothetic or isometric) transformations of M onto itself is called a *conformal transformation* (a *homothetic transformation* or an *isometry*) group and is denoted by $C(M)$ ($H(M)$ or $I(M)$). The connected components of the identity of $C(M)$, $H(M)$ and $I(M)$ are denoted by $C_0(M)$, $H_0(M)$ and $I_0(M)$, respectively.

If a vector field v on M defines an infinitesimal conformal transformation on (M, g), then v satisfies $L_v g = 2\varrho g$, where L_v denotes the Lie derivative with respect to v, and ϱ is a function on M. The vector field v defines an infinitesimal homothetic transformation or an infinitesimal isometry according as ϱ is constant or zero.

In the last decade or so various authors have studied the conditions for a Riemannian manifold M of dimension $n > 2$ with constant scalar curvature R and admitting either an infinitesimal non-isometric or a non-homothetic conformal transformation to be either conformal or isometric to an n-sphere. Very recently Yano, Obata and Sawaki [7]-[9] and Hsiung and Mugridge [4] have been able to extend some of these results by replacing the constancy of R by $L_v R = 0$, where v is a certain vector field on M. The purpose of this paper is to extend further the joint work of Yano and Sawaki [9] and that of Hsiung and Mugridge [4]. Throughout this paper all Latin indices take the values $1, \ldots, n$ unless stated otherwise, and we shall follow the usual tensor convention that indices can be lowered and raised by using, respectively, g_{ij} and g^{ij}, the elements of the inverse of the matrix (g_{ij}), and that repeated indices imply summation.

In the proofs of the theorems in Section 3 we need the following known theorems:

THEOREM A (Ishihara and Tashiro [5], Tashiro [6]). *If a complete Riemannian manifold M of dimension $n \geqslant 2$ admits a non-constant function ϱ such that*

$$\nabla_i \nabla_j \varrho + \frac{1}{n} \Delta \varrho g_{ij} = 0,$$

then M is conformal to an n-sphere.

THEOREM B (Yano and Obata [8]). *A complete Riemannian manifold M of dimension $n \geqslant 2$ is isometric to an n-sphere, if it admits a non-constant scalar field u such that*

$$L_{du} R = 0 \tag{1.1}$$

and

$$\nabla_i u_j + \frac{1}{n} \Delta u g_{ij} = 0 \tag{1.2}$$

hold, where L_{du} is the Lie derivative with respect to the vector field u^i defined by

$$u_j = \nabla_j u, \qquad u^i = g^{ij} u_j, \tag{1.3}$$

∇ being the covariant derivative with respect to g.

2. Notation and formulas. Let (M, g) be a Riemannian manifold of dimension $n \geqslant 2$ and class C^3. In this section we shall list some well-known formulas for our later use; for the details of their derivation see, for example, [1], p. 89-90, where the Riemann tensor differs from ours in sign.

Suppose that (M, g) admits a conformal change of metric

$$g^*_{ij} = e^{2\sigma} g_{ij}, \tag{2.1}$$

where σ is a C^3 function on M. Then

$$g^{*ij} = e^{-2\sigma} g^{ij}. \tag{2.2}$$

For any tensor with respect to g, the corresponding tensor with respect to g^* will be denoted by the same letter with a star.

If we denote by R_{hijk} and R_{ij} the Riemann tensor and the Ricci tensor of (M, g), respectively, we have

$$(2.3)\qquad e^{-2\sigma}R^*_{hijk} = R_{hijk} - g_{hk}\sigma_{ij} - g_{ij}\sigma_{hk} + g_{hj}\sigma_{ik} + g_{ik}\sigma_{hj} - \\ - (g_{hk}g_{ij} - g_{hj}g_{ik})\Delta_1\sigma,$$

where we have put

$$(2.4)\qquad \sigma_{ij} = \nabla_i\nabla_j\sigma - \nabla_i\sigma\nabla_j\sigma, \quad \Delta_1\sigma = \nabla^i\sigma\nabla_i\sigma.$$

By means of equations (2.2) and (2.3), we can have

$$(2.5)\qquad R^*_{ij} = R_{ij} + (2-n)\sigma_{ij} + g_{ij}[\Delta\sigma + (2-n)\Delta_1\sigma],$$

$$(2.6)\qquad R^* = e^{-2\sigma}[R + 2(n-1)\Delta\sigma + (n-1)(2-n)\Delta_1\sigma],$$

where

$$(2.7)\qquad \Delta\sigma = -\nabla^i\nabla_i\sigma.$$

On the manifold M consider the tensor fields

$$(2.8)\qquad T_{ij} = R_{ij} - \frac{R}{n}g_{ij},$$

$$(2.9)\qquad T_{hijk} = R_{hijk} - \frac{R}{n(n-1)}(g_{hk}g_{ij} - g_{hj}g_{ik}),$$

$$(2.10)\qquad W_{hijk} = aT_{hijk} + b_1g_{hk}T_{ij} - b_2g_{hj}T_{ik} + b_3g_{ij}T_{hk} - \\ - b_4g_{ik}T_{hj} + b_5g_{hi}T_{jk} - b_6g_{jk}T_{hi},$$

where the a and b's are constant. (2.10) is the covariant tensor field of order 4 defined by Hsiung [3]. It should be noted that

$$(2.11)\qquad T_{ij} = g^{hk}T_{hijk}, \quad g^{ij}T_{ij} = 0,$$

$$(2.12)\qquad g^{hi}T_{hijk} = 0,$$

$$(2.13)\qquad \nabla^iT_{ij} = \frac{n-2}{2n}\nabla_jR,$$

(2.12) and (2.13) being consequences of the first and second Bianchi identities, respectively.

Substituting (2.3), (2.5) and (2.6) in the equation corresponding to (2.9) for T^*_{hijk}, we obtain, after an elementary simplification,

$$(2.14)\qquad e^{-2\sigma}T^*_{hijk} = T_{hijk} - g_{hk}\sigma_{ij} - g_{ij}\sigma_{hk} + g_{hj}\sigma_{ik} + \\ + g_{ik}\sigma_{hj} - \frac{2}{n}(g_{hk}g_{ij} - g_{hj}g_{ik})(\Delta\sigma + \Delta_1\sigma).$$

Multiplication of both sides of (2.14) by g^{*hk} and use of (2.8), (2.2) and (2.4) give immediately

$$T^*_{ij} = T_{ij} + (n-2)P_{ij}, \tag{2.15}$$

where

$$P_{ij} = -\sigma_{ij} - \frac{1}{n}(\Delta\sigma + \Delta_1\sigma)g_{ij}. \tag{2.16}$$

By substituting (2.14) and (2.15) in the equation corresponding to (2.10) for W^*_{hijk}, we obtain

$$\begin{aligned} &e^{-2\sigma} W^*_{hijk} \\ &= W_{hijk} + a\left[-g_{hk}\sigma_{ij} - g_{ij}\sigma_{hk} + g_{hj}\sigma_{ik} + g_{ik}\sigma_{hj} - \frac{2}{n}(g_{hk}g_{ij} - g_{hj}g_{ik})(\Delta\sigma + \Delta_1\sigma)\right] + \\ &\quad + (n-2)[b_1 g_{hk}P_{ij} - b_2 g_{hj}P_{ik} + b_3 g_{ij}P_{hk} - b_4 g_{ik}P_{hj} + b_5 g_{hi}P_{jk} - b_6 g_{jk}P_{hi}]. \end{aligned} \tag{2.17}$$

On the other hand, if we set

$$u = e^{-\sigma}, \tag{2.18}$$

it follows that

$$\sigma = -\ln u. \tag{2.19}$$

Substituting (2.19) in (2.4) and (2.7) yields immediately

$$\sigma_{ij} = -(\nabla_i u_j)/u, \quad \Delta_1\sigma = (\Delta_1 u)/u^2, \quad \Delta\sigma = -(\Delta u)/u - (\Delta_1 u)/u^2. \tag{2.20}$$

Therefore (2.6), (2.14), (2.16) and (2.17) become, in consequence of (2.20) and (2.4),

$$R^* = u^2 R - 2(n-1)u\Delta u - n(n-1)u_i u^i, \tag{2.21}$$

$$\begin{aligned} u^2 T^*_{hijk} &= T_{hijk} + u^{-1}[g_{hk}\nabla_i u_j + g_{ij}\nabla_h u_k - g_{hj}\nabla_i u_k - \\ &\quad - g_{ik}\nabla_h u_j + \frac{2}{n}(g_{hk}g_{ij} - g_{hj}g_{ik})\Delta u] \\ &= T_{hijk} + g_{hk}P_{ij} + g_{ij}P_{hk} - g_{hj}P_{ik} - g_{ik}P_{hj}, \end{aligned} \tag{2.22}$$

$$P_{ij} = u^{-1}\left(\nabla_i u_j + \frac{\Delta u}{n} g_{ij}\right), \tag{2.23}$$

$$u^2 W^*_{hijk} = W_{hijk} + \frac{1}{u} t_{hijk}, \tag{2.24}$$

where we have placed

$$\begin{aligned} \frac{t_{hijk}}{(n-2)u} &= \left(\frac{a}{n-2} + b_1\right) g_{hk}P_{ij} - \left(\frac{a}{n-2} + b_2\right) g_{hj}P_{ik} + \\ &\quad + \left(\frac{a}{n-2} + b_3\right) g_{ij}P_{hk} - \left(\frac{a}{n-2} + b_4\right) g_{ik}P_{hj} + b_5 g_{hi}P_{jk} - b_6 g_{jk}P_{hi}. \end{aligned} \tag{2.25}$$

From (2.23) and the second equation of (2.11), it follows immediately

$$T_{ij}P^{ij} = u^{-1}T_{ij}\nabla^i\nabla^j u. \tag{2.26}$$

We also have, from (2.15), (2.22) and (2.24), respectively,

$$T^*_{ij}T^{*ij} = u^4[T_{ij}T^{ij} + 2(n-2)T_{ij}P^{ij} + (n-2)^2 P_{ij}P^{ij}], \tag{2.27}$$

$$T^*_{hijk}T^{*hijk} = u^4[T_{hijk}T^{hijk} + 8T_{ij}P^{ij} + 4(n-2)P_{ij}P^{ij}], \tag{2.28}$$

$$W^*_{hijk}W^{*hijk} = u^4\left(W_{hijk}W^{hijk} + \frac{2}{u}t_{hijk}W^{hijk} + \frac{1}{u^2}t_{hijk}t^{hijk}\right). \tag{2.29}$$

Substituting (2.10) and (2.25) in (2.29) and making use of (2.12) and $g^{ij}P_{ij} = 0$, an elementary but lengthy calculation gives

$$W^*_{hijk}W^{*hijk} = u^4[W_{hijk}W^{hijk} + 2c(n-2)T_{ij}P^{ij} + c(n-2)^2 P_{ij}P^{ij}], \tag{2.30}$$

where

$$c = \frac{4a^2}{n-2} + 2a\sum_{i=1}^{4} b_i + \left(\sum_{i=1}^{6}(-1)^{i-1}b_i\right)^2 + \\ + (n-1)\sum_{i=1}^{6} b_i^2 - 2(b_1b_3 + b_2b_4 - b_5b_6). \tag{2.31}$$

An elementary calculation shows that $c \geqslant 0$, where the equality holds if and only if $b_1 = \ldots = b_4$, $b_5 = b_6 = 0$ and $a = (2-n)b_1$.

3. Lemmas.

LEMMA 3.1. *Suppose that a compact orientable Riemannian manifold M admits a conformal change of metric* (2.1). *Then, for an arbitrary number p,*

$$\int_M u^{p-1}T_{ij}u^iu^j dV + \int_M u^{p+1}P_{ij}P^{ij}dV \\ = -\frac{1}{2n}\int_M u^{p-1}(u^{-1}L_{du}R^* - uL_{du}R)dV - \\ -(n+p-2)\left[\int_M u^{p-2}u^iu^j\nabla_i u_j\, dV + \frac{1}{2n(n-1)}\int_M (Ru^{p-1} - R^*u^{p-3})u_iu^i dV - \right. \\ \left. -\frac{1}{2}\int_M u^{p-3}(u_iu^i)^2 dV\right]. \tag{3.1}$$

In particular, if $p = 2-n$, *then*

$$(3.2)\qquad \int_M u^{-n+1} T_{ij} u^i u^j dV + \int_M u^{-n+3} P_{ij} P^{ij} dV$$

$$= -\frac{1}{2n} \int_M u^{-n+1}(u^{-1} L_{du} R^* - u L_{du} R)\, dV.$$

Proof. From (2.23) it follows that

$$(3.3)\qquad u^2 P_{ij} P^{ij} = \nabla_i u_j \nabla^i u^j - (\Delta u)^2/n.$$

Multiplying (3.3) by u^{p-1}, integrating over M, using the equations obtained by directly computing $\nabla^i(u^{p-1} u^j \nabla_i u_j)$ and $\nabla_i(u^{p-1} u^i \Delta u)$, applying the well-known Green's formula

$$(3.4)\qquad \int_M \nabla^i \xi_i dV = 0,$$

ξ_i being any vector field on M, and substituting

$$(3.5)\qquad u^j \nabla^i \nabla_i u_j = R_{ij} u^i u^j - u^i \nabla_i \Delta u,$$

in the resulting equation we obtain

$$(3.6)\qquad \int_M u^{p+1} P_{ij} P^{ij} dV = -(p-1) \int_M u^{p-2} u^i u^j \nabla_i u_j\, dV -$$

$$- \int_M u^{p-1} R_{ij} u^i u^j dV - \frac{n-1}{n} \int_M u^{p-1} u^i \nabla_i \Delta u dV + \frac{p-1}{n} \int_M u^{p-2} u_i u^i dV.$$

On the other hand, solving (2.21) for Δu, we have

$$(3.7)\qquad \Delta u = \frac{1}{2(n-1)} (uR - u^{-1} R^* - n(n-1) u^{-1} u_i u^i).$$

Substituting (3.7) in (3.6), and using (2.8) and

$$(3.8)\qquad L_{du} R = u^i \nabla_i R, \quad L_{du} R^* = u^i \nabla_i R^*,$$

an elementary computation leads readily to the required formula (3.1).

LEMMA 3.2. *Suppose that a compact orientable Riemannian manifold M admits a conformal change of metric* (2.1). *Then*

$$(3.9)\qquad \int_M (u^{-3} \lambda^* - u\lambda)\, dV = -\frac{c(n-2)^2}{n} \int_M L_{du} R dV +$$

$$+ c(n-2)^2 \int_M u P_{ij} P^{ij} dV,$$

$$(3.10)\qquad \int_M (u^{-n+3}\lambda - u^{-n-1}\lambda^*)\,dV = \frac{c(n-2)^2}{n}\int_M u^{-n}L_{du}R^*\,dV + c(n-2)^2\int_M u^{-n+3}P_{ij}P^{ij}\,dV,$$

where

$$(3.11)\qquad \lambda = W_{hijk}W^{hijk},\quad \lambda^* = W^*_{hijk}W^{*hijk},$$

and c *is defined by* (2.31).

Proof. From (2.30) and (2.26) it follows that

$$(3.12)\qquad u^{-3}\lambda^* - u\lambda = c(n-2)(2T_{ij}\nabla^i u^j + (n-2)uP_{ij}P^{ij}).$$

Integrating (3.12) over M, using

$$(3.13)\qquad \nabla^i(T_{ij}u^j) = u^j\nabla^i T_{ij} + T_{ij}\nabla^i u^j,$$

and (2.13), (3.8), and applying Green's formula (3.4), we can easily obtain (3.9).

Similarly, we can derive (3.10) by multiplying (3.12) by u^{-n+2}, integrating the resulting equation over M, using (3.13), (2.13), (3.8), (3.2), and applying Green's formula (3.4).

4. Theorem.

THEOREM 4.1. *If a compact Riemannian manifold* M *of dimension* $n \geqslant 3$ *admits a conformal change of metric* (2.1) *with* $\sigma \leqslant 4$ *such that*

$$(4.1)\qquad L_{du}R^* \geqslant u^n L_{du}R,$$

$$(4.2)\qquad u^\sigma\lambda = \big((u-1)\varphi+1\big)\lambda^*,\quad c>0,$$

hold, where $u = e^{-\sigma}$, φ *is a differentiable non-negative function of* M, *and* c *is given by* (2.31), *then* M *is conformal to an n-sphere. In particular, for the case* $\sigma = 4$ *and* $\varphi = 0$, *we have the same conclusion with condition* (4.1) *replaced by condition*

$$(4.3)\qquad L_{du}R < 0.$$

Finally, M *is isometric to an n-sphere under condition* (1.1), *in addition to conditions* (4.1) *and* (4.2).

Proof. At first we notice

$$(4.4)\qquad \int_M (u\lambda - u^{-3}\lambda^*)\,dV - \int_M (u^{-n+3}\lambda - u^{-n-1}\lambda^*)\,dV = \int_M (u^{n-2}-1)(u^{-n+3}\lambda - u^{-n-1}\lambda^*)\,dV.$$

Furthermore,

$$(4.5)\qquad \int_M (u^{n-2}-1)(u^{-n+3}\lambda - u^{-n-1}\lambda^*)dV - \int_M u^{-n-1}(u^{n-2}-1)(u^{4-\sigma}-1)\lambda^* dV$$

$$= \int_M u^{-n+3-\sigma}(u^{n-2}-1)(u^\sigma\lambda - \lambda^*)dV = \int_M u^{-n+3-\sigma}(u^{n-2}-1)(u-1)\varphi\lambda^* dV \geqslant 0,$$

the last two steps being, respectively, due to (4.2) and

$$u > 0, \quad (u^{n-2}-1)(u-1) \geqslant 0, \quad \varphi \geqslant 0, \quad \lambda^* \geqslant 0.$$

Since $\sigma \leqslant 4$, we have $(u^{n-2}-1)(u^{4-\sigma}-1) \geqslant 0$ so that (4.4) and (4.5) imply

$$(4.6)\qquad \int_M (u\lambda - u^{-3}\lambda^*)dV - \int_M (u^{-n+3}\lambda - u^{-n-1}\lambda^*)dV \geqslant 0.$$

On the other hand, using (3.9), (3.10) and (4.1), we obtain

$$(4.7)\qquad \int_M (u\lambda - u^{-3}\lambda^*)dV - \int_M (u^{-n+3}\lambda - u^{-n-1}\lambda^*)dV$$

$$= \frac{c(n-2)^2}{n}\int_M (L_{du}R - u^{-n}L_{du}R^*)dV - c(n-2)^2\int_M (u + u^{-n+3})P_{ij}P^{ij}dV \leqslant 0.$$

Thus, from $c > 0$ and assumption (4.1), it is readily seen that the equality holds in both (4.6) and (4.7), and, therefore, that

$$(4.8)\qquad \int_M (u + u^{-n+3})P_{ij}P^{ij}dV = 0,$$

which together with (2.23) gives (1.2). Hence, by Theorem A, M is conformal to an n-sphere.

When $\sigma = 4$ and $\varphi = 0$, the left-hand side of (3.9) is reduced to zero and the right-hand side together with condition (4.3) gives immediately (1.2).

Finally, if (1.1) holds, then M is isometric to an n-sphere by the same argument as above and Theorem B, q.e.d.

Theorem 4.1 is due to Yano and Sawaki [9] when

$$(4.9)\qquad a = 0, \quad b_2 = \dots = b_6 = 0$$

or

$$(4.10)\qquad b_1 = \dots = b_4, \quad b_5 = b_6 = 0,$$

and condition (4.1) is replaced by $L_{du}R = L_{du}R^* = 0$. Theorem 4.1 is also due to Hsiung and Mugridge [4] when $\varphi = 0$, $\sigma = 4$ and condition (4.1) is replaced by $\int_M R\Delta u dV = 0$, and due to Yano and Obata [8] with an additional condition (4.9) or (4.10).

REFERENCES

[1] L. P. Eisenhart, *Riemannian geometry*, 2nd ed., Princeton University Press, Princeton 1949.

[2] S. I. Goldberg and K. Yano, *Manifolds admitting a non-homothetic conformal transformation*, Duke Mathematical Journal 37 (1970), p. 655-670.

[3] C. C. Hsiung, *On the group of conformal transformations of a compact Riemannian manifold. III*, Journal of Differential Geometry 2 (1968), p. 185-190.

[4] — and L. R. Mugridge, *Conformal changes of metrics on a Riemannian manifold*, Mathematische Zeitschrift 119 (1971), p. 179-187.

[5] S. Ishihara and Y. Tashiro, *On Riemannian manifolds admitting a concircular transformation*, Mathematical Journal of Okayama University 9 (1959), p. 19-47.

[6] Y. Tashiro, *Complete Riemannian manifolds and some vector fields*, Transactions of the American Mathematical Society 117 (1965), p. 251-275.

[7] K. Yano, *On Riemannian manifolds admitting an infinitesimal conformal transformation*, Mathematische Zeitschrift 113 (1970), p. 205-214.

[8] — and M. Obata, *Conformal changes of Riemannian metrics*, Journal of Differential Geometry 4 (1970), p. 53-72.

[9] K. Yano and S. Sawaki, *Notes on conformal changes of Riemannian metrics*, Kōdai Mathematical Seminar Reports 22 (1970), p. 480-500.

LEHIGH UNIVERSITY
LAFAYETTE COLLEGE

Reçu par la Rédaction le 10. 12. 1971